The Atiyah - Singer Index Theorem

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Abstract

In this paper we will give an overview of the work needed to define the analytical and topological indices and state the Atiyah - Singer index theorem. We will then give an explanation of how the Atiyah - Singer index theorem can be used to prove the classical Riemann - Roch theorem. For the sections on the analytical index we referred to [BB85] and [Gil95]. For the sections on the topological index we referred primarily to [ASI], but additional sources such as [Atiyah] and [Segal] are useful.

0 Introduction

The study of partial differential equations can be viewed, broadly speaking, as the study of partial differential operators P on some manifold X. Many standard questions in PDEs can be reformulated in terms of algebraic properties of the operator P. The kernel of P is the set of solutions to the equation $Pf = 0$, while the cokernel of P gives the constraints a function g must satisfy in order for the equation $Pf = g$ to have solutions. It would therefore be incredibly useful to be able to compute $\ker P$ and $\operatorname{cok} P$ or at least $\dim \ker P$ and $\dim \operatorname{cok} P$ for an arbitrary partial differential operator P. This turns out to be an extremely difficult question in general because $\ker P$ and $\operatorname{cok} P$ are not stable under perturbations of P in any reasonable sense. If however P is an elliptic differential operator, then the quantity $a\text{-ind } P := \dim \ker P - \dim \operatorname{cok} P$, the analytical index of P, is well-defined and stable under perturbations. This means that the analytical index is a topological quantity and raises the question of whether the index could be computed using purely topological methods. This question was answered in the affirmative by Atiyah and Singer [ASI] who constructed a topological index map and showed that it agreed with the analytical index. The Atiyah-Singer index theorem is a generalization many other theorems relating analytical and topological data, namely the Gauss-Bonnet, Riemann-Roch, and Hirzebruch-Riemann-Roch theorems.

Furthermore, Atiyah and Singer proved the index theorem in two ways, each aligning with a generalization of Riemann-Roch. Originally, they proved the index theorem analogously to Hirzebruch's proof of Riemann-Roch using cobordism. In [ASI], they explain that the proof presented in this series of papers aligns closer to Grothendieck's proof of Riemann-Roch using K-theory. This connection is best understood with cohomological interpretations of the topological index, which can be found in [ASIII]. As such, this can be viewed as a real analog of Grothendieck-Riemann-Roch, which is naturally stated for complex manifolds. The theorem was later reproven by Atiyah, Bott, and Patodi [ABP73] using a more analytical heat kernel approach.

1 Notation

We begin by fixing some notation. For a multi-index $\alpha = (\alpha_1, ..., \alpha_d)$, we let $|\alpha| = \alpha_1 + ... + \alpha_d$ and $D_x^\alpha := (-i)^{|\alpha|}\partial_x^\alpha := (-i)^{|\alpha|}\frac{\partial^{\alpha_1}}{\partial(x_1)^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial(x_d)^{\alpha_d}}$. For a Schwartz function f, its Fourier transform given by $\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \int e^{-ix\cdot\xi} f(x)dx$, where dx is Lebesgue measure scaled by a factor of $(2\pi)^{-d/2}$. The inverse Fourier transform is then given by $\check{f}(x) := \mathcal{F}^{-1}(f)(x) := \int e^{ix\cdot\xi} f(\xi)d\xi$, with $d\xi$ scaled as above. These normalizations are chosen so that $D_x^\alpha f = \mathcal{F}^{-1}(\xi^\alpha \hat{f}(\xi))$.

2 Differential and Pseudodifferential Operators on $\mathbb{R}^d$

The statement of the Atiyah-Singer index theorem concerns differential operators between vector bundles over a Riemannian manifold, but we will begin with the simpler case of differential operators on $\mathbb{R}^d$ (where both vector bundles are the trivial one dimensional bundle).

Definition 2.1. A *differential operator* of order k on $\mathbb{R}^d$ is a linear map $P : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ of the form

$$P := \sum_{|\alpha| \leq k} a_\alpha(x) D_x^\alpha,$$

where for each multi-index α, $a_\alpha \in C^\infty(\mathbb{R}^d)$. We associate to P a *symbol* $\sigma(P)$ which is a function of (x, ξ) given by

$$\sigma(P)(x, \xi) := \sum_{|\alpha| \leq k} a_\alpha(x) \xi^\alpha.$$

The symbol of P describes the behaviour of P in Fourier space: for any Schwartz function f we have

$$Pf(x) = \int e^{ix\cdot\xi} \sigma(P)(x, \xi) \hat{f}(\xi) d\xi.$$

A classic example of a differential operator is the Laplacian Δ whose symbol is $(-1)^d |\xi|^2$. One classical method to solve the equation $\Delta f = g$ for a fixed Schwartz function g is to take a Fourier transform, giving $(-1)|^d |\xi|^2 \hat{f} = \hat{g}$. This lets us (at least formally) recover f as the inverse Fourier transform of $(-1)^d |\xi|^{-2} \hat{g}$, i.e.

$$f(x) = \int e^{ix\cdot\xi} (-1)^d |\xi|^{-2} \hat{g}(\xi) d\xi.$$

The key step here was that we could invert the symbol of Δ (away from $\xi = 0$). It would seem useful then to consider the class of differential operators P whose symbol $\sigma(P)$ is invertible. However we will soon want to consider differential operators on manifolds, where it turns out that the full symbol does not transform properly under change of coordinates. If we instead look at the leading terms of the symbol then this transforms like a rank k tensor.

Definition 2.2. If

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) D_x^\alpha$$

is a differential operator of order k then its *leading symbol* is given by

$$\sigma_L(P) := \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha.$$

Unfortunately if $\sigma_L(P)$ is invertible (we could just say nonzero here but later $\sigma_L(P)$ will be a linear map) that does not mean that $\sigma(P)$ need be invertible and so our trick above to find an inverse for P fails. We do however know that $\sigma_(P)$ will be invertible for sufficiently large ξ and this, it turns out, will be enough to construct a "pseudo-inverse" for P, as we shall see later.

Definition 2.3. A differential operator P is elliptic if $\sigma_L(P)$ is invertible for all $\xi \neq 0$.

As the above example shows, when P is a differential operator, $\sigma(P)^{-1}$ is almost never the symbol of another differential operator. Thus to find inverse operators, we must expand our class of allowable symbols. We will define pseudodifferential operators to be those operators corresponding to this larger symbol class. In order to ensure that pseudodifferential operators have the same analytical properties as differential operators of the same order, we will need to ensure that our new symbols satisfy the same asymptotic growth bounds in ξ.

Definition 2.4. The class Symb^k of *symbols of order* k consists of all smooth functions $p(x,\xi)$ with compact support in x satisfying

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \lesssim_{\alpha,\beta} (1+|\xi|)^{m-|\beta|}.$$

To every symbol $p \in \mathrm{Symb}^k$, we associate a *pseudodifferential operator* P with symbol p of order k via

$$Pf(x) := \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi.$$

Note that this definition makes sense for any $k \in \mathbb{R}$ whereas differential operators of order k only make sense when $k \in \mathbb{N}$. In the rest, however, we will only need pseudodifferential operators of order $k \in \mathbb{Z}$. At the moment, we will leave the domain and codomain of the pseudodifferential operator ambiguous, but we will clarify this more when we define the analytical index.

3 Differential and Pseudodifferential Operators on Manifolds

To define differential and pseudodifferential operators on a manifold we must work locally. We fix a smooth manifold X of dimension d.

Definition 3.1. A (scalar) differential operator on X is a linear operator $P : C^\infty(X) \to C^\infty(X)$ such that for every chart (O,h) for X and every $\phi,\psi \in C_c^\infty(O)$ the pushforward $h_*(\phi P \psi)$ is a (scalar) differential operator on $h(O)$, i.e. it has the form

$$h_*(\phi P\psi) = \sum_{|\alpha|\le k} a_\alpha(x) D_x^\alpha$$

for some functions $a_\alpha \in C_c^\infty(h(O))$. We likewise say P is a (scalar) pseudodifferential operator on X if $h_*(\phi P \psi)$ is a pseudodifferential operator on $\mathbb{R}^d$ whose symbol has x-support compactly contained in $h(O)$.

So far we have considered only differential and pseudodifferential operators acting on scalar-valued functions. Many classical differential operators that arise in geometry are not of this form however. One example is the exterior derivative which, for a given smooth manifold X, is a differential operator from $\Omega^k(X)$ to $\Omega^{k+1}(X)$. Given vector bundles E and F over X, we define differential and pseudodifferential operators from E to F in an analogous way by working with local trivializations of E and F and letting the functions $a_\alpha(x)$ be functions in $C^\infty(\hom(E,F))$. We will omit the details of this definition for the sake of brevity. The key point we will need about this definition is that for a given point $(x,\xi) \in T^*X$, the leading symbol $\sigma_L(P)$ gives a map between the fibers E_x and F_x above x. Thus if $\pi : T^*X \to X$ is the projection map, then $\sigma_L(P)$ is a bundle homomorphism from $\pi^*(E)$ to $\pi^*(F)$. If P is an elliptic differential operator then $\sigma_L(P)$ is an isomorphism away from the zero section.

4 Fredholm Operators

If $T : V_1 \to V_2$ is a linear map between finite dimensional vector spaces V_1 and V_2, then there is little in general that can be said about either the kernel of T or the image of T if we consider them separately. We do, however, have the rank-nullity theorem which tells us that $\dim \operatorname{im} T + \dim \ker T = \dim V_1$. Alternatively, if we define $\operatorname{ind} T := \dim \ker T - \dim \operatorname{cok} T$, then the rank nullity theorem says that $\operatorname{ind} T = \dim V_1 - \dim V_2$. We will explore in this section what can be said about $\operatorname{ind} T$ when we replace V_1 and V_2 with infinite dimensional Hilbert spaces H_1 and H_2. The first obstacle is of course that $\operatorname{ind} T$ need not be defined - if $\operatorname{im} T$ is not closed then $\operatorname{cok} T$ does not make sense as a Hilbert space and moreover $\ker T$ and $\operatorname{cok} T$ may be infinite dimensional. This motivates the following definition.

Definition 4.1. Let $T : H_1 \to H_2$ be a bounded operator. We say that T is *Fredholm* if $\operatorname{im} T$ is closed and $\dim \ker T, \dim \operatorname{cok} T < \infty$. In this case we define the *Fredholm index* of T to be

$$\operatorname{ind} T := \dim \ker T - \dim \operatorname{cok} T.$$

We denote by $\operatorname{Fred}(H_1, H_2)$ the set of all Fredholm operators.

Note that we can identify $\operatorname{cok} T$ with $\ker T^*$ and so we could alternatively define $\operatorname{ind} T = \dim \ker T - \dim \ker T^*$. Any invertible operator is trivially Fredholm with index 0. The simplest nontrivial examples of a Fredholm operator is the unilateral shift map $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $(a_1, a_2, a_3, ...) \mapsto (0, a_1, a_2, ...)$ and its adjoint $S^* : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $(a_1, a_2, a_3, ...) \mapsto (a_2, a_3, a_4, ...)$. The shift map has index 1 while its adjoint has index -1 and moreover powers of S and S^* give us examples of Fredholm operators of each index. These examples generalize as follows:

Proposition 4.1.

(i) If T is Fredholm then so is T^* and $\operatorname{ind} T^* = -\operatorname{ind} T$.

(ii) If $T : H_1 \to H_2$ and $S : H_2 \to H_3$ are Fredholm then so is $ST : H_1 \to H_3$ and $\operatorname{ind} ST = \operatorname{ind} T + \operatorname{ind} S$.

The key benefit of working with the index is that it is a stable quantity. Given an operator T, it is possible to find a small perturbation T' of T with wildly different kernel and cokernel, but the index will be unchanged. In order to state this more precisely, we need to define what we mean by a "small perturbation".

Definition 4.2. Let $T : H_1 \to H_2$ be a bounded linear operator. We say that T is *finite rank* if $\dim \operatorname{im} T < \infty$. We say that T is *compact* if T is the norm-limit of finite rank operators.

The term compact comes from an alternative characterization of compact operators, namely that T is compact if and only if the image of the unit ball under T is relatively compact in H. The notion of compact operators let's us give an alternative characterization of the class of Fredholm operators.

Proposition 4.2. *A bounded operator $T : H_1 \to H_2$ is Fredholm if and only if it is invertible modulo compact operators, i.e. there exists $S : H_2 \to H_1$ such that $TS - I_{H_1}$ and $ST - I_{H_2}$ are compact, where I_{H_1} and I_{H_2} are the identity operators on H_1 and H_2 respectively.*

We also have stability of the index with respect to compact perturbations.

Proposition 4.3. *If $T : H_1 \to H_2$ is Fredholm and $K : H_1 \to H_2$ is compact then $T + K$ is Fredholm and* $\operatorname{ind}(T + K) = \operatorname{ind} T$.

In the case that $H_1 = H_2 = H$, there is a nice picture of what is going. If we let $\mathcal{B}(H)$ and $\mathcal{K}(H)$ denote the bounded and compact operators on H respectively (and view them as C^*-algebras), then we have a short exact sequence

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow \mathcal{B}(H) \longrightarrow \mathcal{Q}(H) \longrightarrow 0.$$

The quotient $\mathcal{Q}(H)$ is called the Calkin algebra and the Fredholm operators, denoted $\operatorname{Fred}(H)$, are precisely the preimage of the invertibles in the Calkin algebra under the projection $\mathcal{B}(H) \to \mathcal{Q}(H)$.

Lastly, and most importantly, the index is stable under norm perturbations.

Proposition 4.4. *The index map* $\operatorname{ind} : \operatorname{Fred}(H_1, H_2) \to \mathbb{Z}$ *is locally constant with respect to the norm topology.*

5 The Analytical Index of an Elliptic Differential Operator

Given an elliptic pseudodifferential operator P, we would like to define its analytical index to be the Fredholm index of P, but to do this we must be able to view P as a bounded linear operator between some Hilbert spaces. To this end, we will need to introduce the Sobolev spaces $H_s, s \in \mathbb{R}$.

Definition 5.1. For a given $s \in \mathbb{R}$ we define the *Sobolev norm* $||\cdot||_s$ on Schwartz functions via

$$||f||_s^2 = \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi.$$

We define the *Sobolev space* H_s to be the completion of the space of Schwartz functions with respect to $||\cdot||_s$.

Sobolev spaces are Hilbert spaces since they are isomorphic to an L^2 space with a weighted Lebesgue measure. We can also define $H_s(X)$ for a Riemannian manifold X by working in charts. Although we need to introduce a Riemannian metric to our smooth manifold X in order to make this construction, this can always be done by a partition of unity argument and the exact metric chosen will have no impact on any of the subsequence development. We will thus assume from now on that all our manifolds come with a Riemannian metric if needed.

The following fact about Sobolev spaces will be key to showing that elliptic differential operators are Fredholm.

Proposition 5.1. *If $s < t$ then $H_t \subseteq H_s$ and the inclusion $\iota_{t,s} : H_t \to H_s$ is compact.*

Proposition 5.2. *Let P be a pseudodifferential operator of order k on X. Then P extends to a bounded linear operator from $H_s(X)$ to $H_{s-k}(X)$ for all $s \in \mathbb{R}$.*

Proof. We will just do the $\mathbb{R}^d$ case since the case for X involves only working in charts. Let $p(x, \xi)$ be the symbol of P so that for any Schwartz function f we have

$$Pf(x) = \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi) d\xi.$$

Then since $|p(x,\xi)| \lesssim (1+|\xi|)^k$ we have

$$\begin{aligned}||Pf||_{s-k}^2 &= \int (1+|\xi|^2)^{s-k}|\widehat{Pf}(\xi)|^2 d\xi \\ &= \int (1+|\xi|^2)^{s-k}|p(x,\xi)\hat{f}(\xi)|^2 d\xi \\ &\lesssim \int (1+|\xi|^2)^{s}|\hat{f}(\xi)|^2 d\xi \\ &= ||f||_s^2,\end{aligned}$$

showing that P extends continuously from H_s to H_{s-k}. ☐

The following proposition tells us that elliptic differential operators are Fredholm, allowing us to define the analytical index of P to be the Fredholm index. We will state it in terms of scalar differential operators, but the appropriate generalization to differential operators between vector bundles holds as well.

Proposition 5.3. *Let P be an elliptic differential operator of order m on X. Then $P : H_s(X) \to H_{s-m}(X)$ is Fredholm.*

Proof. We give a very rough sketch of the idea of the proof here. To show that P is Fredholm, we must construct a pseudoinverse $Q : H_{s-m} \to H_s$ so that $PQ - I_{H_{s-m}}$ and $QP - I_{H_s}$ are compact. Intuitively, since $\sigma_L(P)$ is invertible for all $\xi \neq 0$, $\sigma(P)$ is invertible for sufficiently large ξ and so we can write $\sigma(P) = p + r$ where $p(x,\xi)$ is invertible for all $\xi \neq 0$ and r has compact ξ-support. If we then let Q be the pseudodifferential operators with symbols p^{-1}, then Q will be the inverse of P modulo a pseudodifferential operator R whose symbol has compact ξ-support. We call such an operator an infinitely smoothing operator since $Rf \in C^\infty(X)$ for all $f \in H_s$. Since $C^\infty(X) \subseteq H_{s+1}$ we can compose R with the inclusion $\iota_{s+1,s} : H_{s+1} \to H_s$ to see that $R = \iota_{s+1,s} \circ R$ is compact. Of course to do this properly we must work in charts since the full symbol of P is not globally defined, but this is nevertheless the approximate idea of the proof. ☐

We would like now to define the analytical index of P to be its Fredholm index as a map from H_s to H_{s-k}, but this need not a priori be independent of s. It might be the case that $\ker P$ or $\ker P^*$ contains non-smooth functions which are in H_s for only some s. This issue is resolved by the following proposition, known as elliptic regularity which says that the solution to any elliptic PDE is smooth.

Proposition 5.4. *Let P be an elliptic differential operator of order m on X. Then for any $s \in \mathbb{R}$ we have that $\ker P \subseteq C^\infty(X)$ when viewing P as a operator from $H_s(X)$ to $H_{s-m}(X)$. In particular $\ker P$ is independent of s.*

Proof. Following the proof of the above proposition, we can find $Q : H_{s-m}(X) \to H_s(X)$ such that $R = QP - I_{H_s}$ is an infinitely smoothing operator. Then for any $f \in H_s(X)$ with $Pf = 0$ we have $f = QPf + Rf = Rf$ and so $f \in C^\infty(X)$. ☐

Definition 5.2. If P is an elliptic differential operator between vector bundles E and F over X then we define its analytical index a-ind P to be its index as a Fredholm operator between Sobolev spaces.

6 Topological K - Theory

We proceed with a brief overview of topological K - theory. This is summarized in [ASI]. A more comprehensive resource is [Atiyah].

6.1 Preliminary definitions

Convention. All spaces going forwards will be Hausdorff and locally compact. This is not strictly necessary for the definitions at hand, but we are ultimately interested in the case of a compact manifold, and are wholly disinterested in the rudiments of point set topology.

Definition 6.1. Let X be a compact topological space. The set of isomorphism classes of complex vector bundles on X forms a commutative semiring under the direct sum as addition and the tensor product as multiplication. The additive group completion therefore yields a commutative ring generated by isomorphism classes of vector bundles, which we denote by $K(X)$. For a vector bundle $E \longrightarrow X$ we let $[E]$ denote its class in K - theory, i.e. in $K(X)$.

Definition 6.2. Let $f : X \longrightarrow Y$ be continuous. Then for a vector bundle E on Y we can associate a vector bundle f^*E on X via the pullback of topological spaces. This extends to a ring homomorphism $f^* : K(Y) \longrightarrow K(X)$.

Note critically the variance of this construction. As such, we have situated K - theory as a contravariant functor from the category of compact spaces to the category of commutative rings. K - theory fits into a general notion in algebraic topology called a cohomology theory. We will not explore this notion in this report, nor will we attempt to discuss the higher K groups. However, many of the following definitions will be familiar to those acquainted with singular cohomology, and are really situated in the notion of cohomology theories.

For one, we have the following immediate computation $K(*) = \mathbb{Z}$. where $*$ is a one point space, as vector bundles on $*$ are the same as finite dimensional vector spaces.

Definition 6.3. Let (X, x) be a based compact space. The reduced K - theory of (X, x) is $\widetilde{K}(X) = \ker(K(X) \longrightarrow K(x))$ given via pulling back the inclusion of the base point.

Note that as $\mathbb{Z}$ is a free abelian group, we attain the (noncanonical) splitting $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$.

Definition 6.4. Let X be a locally compact space. We can form its one point compactification X^+. As such, we define its K - theory as $K(X) = \widetilde{K}(X)$.

Remark. Note critically that not every continuous map $X \longrightarrow Y$ between locally compact spaces induces a map $X^+ \longrightarrow Y^+$. It is only the proper maps $X \longrightarrow Y$ which do this, so functoriality for K - theory on locally compact spaces is restricted to proper maps. As such, we think of this as K - theory with compact support.

6.2 Functoriality

We have seen above that there is a natural pullback morphism on K - theory. This is typical of cohomology theories, but to proceed with the topological index we will need two instances of functoriality "going the wrong way".

First, we seek a pushforward along open inclusions.

Proposition 6.1. *Let U be an open subset of a locally compact space X and let $i : U \longrightarrow X$ be the inclusion. There is a pushforward on K - theory $i_* : K(U) \longrightarrow K(X)$. In fact, K - theory is continuous in the sense that the induced map*

$$\operatorname*{colim}_{U \subseteq X \text{ open}} K(U) \longrightarrow K(X)$$

is an isomorphism

Proof. We provide the construction of i_*. Observe that $X^+/(X^+ - U)$ is a compact space consisting of U and one additional point corresponding to $X^+ - U$. As such, $X^+/(X^+ - U) = U^+$ so we have formed a map $X^+ \longrightarrow U^+$. The pushforward i_* is then defined subsequently as the pullback along this map. □

The next covariant functoriality we seek is considerably more complex.

Proposition 6.2. *Let i be the inclusion of a compact manifold X into another manifold Y. There is a map $i_! : K(TX) \longrightarrow K(TY)$.*

This map is pronounced "i shriek", or perhaps "i lower shriek" in some contexts. Note that this is a map between K - theory on the tangent bundles of these manifolds, not the manifolds themselves. Indeed, we will see below that elliptic differential operators on X have a natural interpretation as bundles on TX, hence our interest in studying them.

Our task now is to construct this map. To do so, we will need to understand the structure of K - theory of E for $E \longrightarrow X$ a real vector bundle. We may then define the Thom map on K - theory, from which the shriek construction will follow.

6.3 K - theory via complexes

Definition 6.5. Let

$$E^\bullet = \ 0 \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^k \longrightarrow 0$$

be a chain complex of vector bundles over X. We say $E^\bullet$ has compact support if the set of points x for which $E^\bullet_x$ is not exact is compact in X.

Let $C^\bullet_c(X)$ be the set of compactly supported chain complexes of vector bundles over X in non-negative degrees and bounded above.

This notion of support is the support of the homology of the sequence. Furthermore, the support is closed as the rank of a map between vector bundles is lower semicontinuous and the nullity is upper semicontinuous.

The direct sum and tensor product of chain complexes turns $C^\bullet_c(X)$ into a semiring. Recall that the tensor product of chain complexes is given by

$$(E^\bullet \otimes F^\bullet)^k = \bigoplus_{i+j=k} E^i \otimes F^j$$

for $E^\bullet, F^\bullet$ be complexes over X.

Definition 6.6. Let $E^\bullet$ and $F^\bullet$ be chain complexes over X. We say that they are homotopic if there is a chain complex $H^\bullet$ over $X \times I$ so that $H^\bullet|_{X\times 0} = E^\bullet$ and $H^\bullet|_{X\times 1} = F^\bullet$.

We let $\overline{C^\bullet_c}(X)$ be the set of homotopy classes of compactly supported chain complexes over X.

Direct sum and tensor product are well defined on homotopy classes. Furthermore, the homotopy relation allows direct sum to become a group operations, so that this is a ring. In fact, the following is true.

Proposition 6.3. *Let X be compact. Then there is a ring isomorphism $\overline{C_c^\bullet}(X) \longrightarrow K(X)$ given by*

$$E^\cdot \mapsto \sum_{i \geq 0} (-1)^i [E^i]$$

In fact, such an isomorphism exists for X locally compact, but we do not include its definition here. Going forwards, we will refer to elements of $K(X)$ using either vector bundles $[E]$ or chain complexes $[E^\bullet]$.

When connecting the K - theory of the (co)tangent bundle of X to elliptic differential operators on X, it will become necessary to find simple representatives of the K - theory of vector bundles over X. Take indeed some real vector bundle $\pi : E \longrightarrow X$.

Definition 6.7. Let E^0, E^1 be complex vector bundles over X. Let $\alpha : \pi^* E^0 \longrightarrow \pi^* E^1$, a morphism of vector bundles over E. Think of this as an element of the Hom bundle $\mathrm{Hom}_E(\pi^* E^0, \pi^* E^1)$. We have the fiber computation $(\pi^* E^i)_e = E^i_{\pi(e)}$. Let $\lambda \in \mathbb{R}$. Then α_e and $\alpha_{\lambda e}$ define elements of $\mathrm{Hom}(E^0_{\pi(e)}, E^1_{\pi(e)})$. We say that α is positively homogeneous of degree k if for all $\lambda > 0$ we have $\alpha_{\lambda e} = \lambda^k \alpha_e$.

With this definition, we can state our desired representation of vector bundles over E.

Proposition 6.4. *Every element of $K(E)$ can be represented by a compactly supported homogeneous complex*

$$0 \longrightarrow \pi^* E^0 \xrightarrow{\alpha} \pi^* E^1 \longrightarrow 0$$

and we refer to this as $[\alpha] \in K(E)$.

Notice that this is the same as saying we have a bundle map $\alpha : \pi^* E^0 \longrightarrow \pi^* E^1$ over E which is an isomorphism away from a compact neighborhood. This is exactly the sort of data the symbol of an elliptic differential operator yields, which provides the basis for our connection between the analytical and topological indices. In particular, elliptical differential operators yield elements in $K(T^*X)$, which we implicitly identify with $K(TX)$ using some Riemannian structure.

6.4 The Thom map

The definition of K - theory via compactly supported complexes affords us additional flexibility in our ability to define elements and maps of K - theory. For instance, we have the following enrichment of the algebraic structure of K - theory.

Proposition 6.5. *Let E be a real vector bundle over X. We have a map*

$$K(X) \otimes K(E) \longrightarrow K(X \times E) \longrightarrow K(E)$$

making $K(E)$ into a $K(X)$ module.

Proof. The first map is given as follows. Let $V^\bullet$, $W^\bullet$ be compactly supported complexes over X and E respectively. We define their external direct product to be

$$V^\bullet \boxtimes W^\bullet = p_X^* V^\bullet \otimes p_E^* W^\bullet$$

which is now a compactly supported complex over the product $X \times E$. Hence, we have the map $K(X) \otimes K(E) \longrightarrow K(X \times E)$ via $[V^\bullet] \otimes [W^\bullet] \mapsto [V^\bullet \boxtimes W^\bullet]$.

To describe the second map, consider first the pullback square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X \times E \arrow[d] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

where Δ is the diagonal map, which notably is proper. As such, take a vector bundle F over $X \times E$ and consider the pullback

$$\begin{tikzcd} \Delta^* F \arrow[r] \arrow[d] & F \arrow[d] \\ E \arrow[r] \arrow[d] & X \times E \arrow[d] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

so we let $F \mapsto \Delta^* F$ be the map $K(X \times E) \longrightarrow K(E)$. □

Definition 6.8. Let E be a complex vector bundle over X. We define the Thom map $\phi_E : K(X) \longrightarrow K(E)$ via $[F] \mapsto [F][\Lambda(E)]$, where $\Lambda(E)$ is the exterior algebra of E.

We have compatibility of the Thom maps via the natural isomorphism $\Lambda(E \otimes F) = \Lambda(E) \otimes \Lambda(F)$. Indeed, the following diagram commutes:

$$\begin{tikzcd} K(X) \arrow[r, "\phi_E"] \arrow[dr, "\phi_{E \oplus F}"'] & K(E) \arrow[d, "\phi_F"] \\ & K(E \oplus F) \end{tikzcd}$$

for complex vector bundles E and F over X.

The key theorem for the Thom map is that it is an isomorphism

Theorem 6.1 (Thom isomorphism theorem/Bott periodicity)**.** *ϕ_E is an isomorphism.*

6.5 The construction of $i_!$

Now we may return to our original question of defining $i_! : K(TX) \longrightarrow K(TY)$ for $i : X \longrightarrow Y$ the inclusion of a compact submanifold of Y. We proceed as follows.

(i) We form a tubular neighborhood N about X in Y. That is, N is an open subset of Y which contains X and admits a retract $\pi : N \longrightarrow X$ to the inclusion. These can be constructed via Riemannian metrics. The tubular neighborhood theorem ensures us that $\pi : N \longrightarrow X$ is isomorphic to the normal bundle of X in Y.

(ii) Apply the tangent bundle functor to $X \subseteq N \subseteq Y$. We will then have $TX \subseteq TN \subseteq TY$ with TN a tubular neighborhood of TX in TY.

(iii) Let $\pi : TX \longrightarrow X$. We seek to identity TN as $\pi^*(N \otimes \mathbb{C})$. We will explain this in the case that X is a point, which provides the fiberwise basis for this identification.

Indeed, for illustration let's say $X = \{0\}$ is contained in $Y = \mathbb{R}$. More general Y are no more challenging from an abstract setting, but $Y = \mathbb{R}$ works best pictorially. For instance, it affords us a concrete normal bundle $N = (-1, 1)$ with the constant map $N \longrightarrow \{0\}$. We have then that $TX = \{0\} \times \mathbb{R}$, $TN = (-1, 1) \times \mathbb{R}$, and $TY = \mathbb{R} \times \mathbb{R}$. Then we can see TN corresponds to $\pi^*(N \oplus N)$, where the first N is the horizontal direction in Y and the second N is the tangent direction. And of course, $N \oplus N = N \oplus iN = N \otimes \mathbb{C}$, so we are viewing the imaginary coordinate as the tangent direction. In this case, TN is indeed $\pi^*(N \otimes \mathbb{C})$.

As discussed, replacing Y by any $\mathbb{R}^n$ is no significant barrier except to our visual intuition, and if we work locally, $Y = \mathbb{R}^n$ is sufficient. Furthermore, reducing from X a compact manifold to X a point comes down to working fiberwise, which is standard fare for working with vector bundles. So we state now that TN is naturally identified with $\pi^*(N \otimes \mathbb{C})$ in the general case.

(iv) We have therefore identified TN as a complex vector bundle on TX, so we have the Thom map $K(TX) \longrightarrow K(TN)$. Furthermore, TN is an open submanifold of TY, so we have the pushforward $K(TN) \longrightarrow K(TY)$. As such, we form the composition $K(TX) \longrightarrow K(TY)$.

Definition 6.9. $i_! : K(TX) \longrightarrow K(TY)$ is the composition of the Thom map $K(TX) \longrightarrow K(TN)$ and the open pushforward $K(TN) \longrightarrow K(TY)$.

Let's remark now on a few notable properties of $i_!$.

(i) The shriek $i_!$ is functorial in the sense that $(ji)_! = j_! i_!$ due to the above compatibility of the Thom map.

(ii) Say X is a point. We can then take the normal bundle N to be all of Y itself. Thus, $i_! : K(TX) \longrightarrow K(TY)$ in this case is precisely the Thom map $\phi_{T(Y)} : K(TX) \longrightarrow K(TY)$.

7 The Topological Index via K - theory

The shriek map defined in the above section is the key to our K - theoretic definition of the topological index. Let X be a smooth manifold and TX its tangent bundle. We seek a morphism $K(TX) \longrightarrow \mathbb{Z}$. Indeed, first consider an embedding $i : X \longrightarrow \mathbb{R}^n$. Additionally, let $j : \{0\} \longrightarrow \mathbb{R}^n$ be the inclusion. We have the shriek maps

$$\begin{tikzcd} K(TX) \arrow[r, "i_!"] & K(T\mathbb{R}^n) \\ & K(T*) \arrow[u, "j_!"'] \end{tikzcd}$$

By the above, $j_!$ is just the Thom map. As such, $j_!$ is invertible by the Thom isomorphism theorem. Furthermore, $K(T*) = K(*) = \mathbb{Z}$. This can be viewed as an instance of the Thom isomorphism theorem for $X = *$ and $E = T*$. As such, we have the diagram

$$\begin{tikzcd} K(TX) \arrow[r, "i_!"] \arrow[ddr, "t-\mathrm{ind}"'] & K(T\mathbb{R}^n) \\ & K(*) \arrow[u, "j_!"', "\sim"] \arrow[d, "\sim"'] \\ & \mathbb{Z} \end{tikzcd}$$

defining the topological index $t - \mathrm{ind}$.

7.1 Characterizing $t - \mathrm{ind}$ axiomatically

We now present a set of axioms which characterize this map $t - \mathrm{ind} : K(TX) \longrightarrow \mathbb{Z}$. However, the axiomatic approach provided in [ASI] proceeds in the more general context of equivariant K - theory, yet we have only provided exposition on standard K - theory. That is, there is some underlying compact Lie group G which we assume to be acting on our spaces and bundles. We assume also that all our maps are G - equivariant. The above discussion took place in the case of $G = 1$ the trivial group.

The only real difficulty in modifying what we have done to the equivariant setting comes down to finding a G - equivariant embedding of X into some finite dimensional representation of G. One can refer to [Segal] for an introduction to equivariant K - theory. Additionally, $K_G(*)$ is no longer $\mathbb{Z}$, rather, it is the representation ring $R(G)$.

The ability to generalize the proof of the index theorem to the equivariant setting is a major strength of Atiyah and Singer's approach. In our context, however, this generality is mostly excess baggage which we do not seek to carry around other than in stating these axioms. The benefit of presenting these axioms is to provide a method by which the theorem can be proven – one needs "only" show that the analytical index satisfies these axioms. Uniqueness is also a clear consequence of the axioms, which will in one fell swoop eliminate any question of the dependence of the topological index on the choices made. As such, stating these axioms is useful enough that we find it warrants the brief foray into the equivarant setting.

Definition 7.1. An *index function* is a collection of $R(G)$ morphisms $t - \mathrm{ind}_G^X : K_G(TX) \longrightarrow R(G)$ for every compact Lie group G and compact G - manifold X which satisfies the following conditions:

(i) $t - \mathrm{ind}_G^X$ is functorial in the variable G.

(ii) $t - \mathrm{ind}_G^X$ is functorial in the variable X under G - diffeomorphisms.

(iii) $t - \mathrm{ind}_G^* : K_G(T*) \longrightarrow R(G)$ is the identity.

(iv) Let $i : X \longrightarrow Y$ be an inclusion of compact G - manifolds. Then the following diagram commutes:

$$\begin{tikzcd} K_G(TX) \arrow[rr, "i_!"] \arrow[dr, "t-\mathrm{ind}_G^X"'] & & K_G(TY) \arrow[dl, "t-\mathrm{ind}_G^Y"] \\ & R(G) & \end{tikzcd}$$

Proposition 7.1. *The method described above, upon appropriate generalization to the equivariant setting, provides an index function. Furthermore, this is the only index function.*

A number of other axioms for which this uniqueness and existence hold are described [ASI, §4].

8 The Atiyah-Singer Index Theorem

8.1 Atiyah - Singer

Given an elliptic differential operator P between vector bundles E and F over X, its leading symbol $\sigma_L(P)$ can be viewed as an element $[\sigma_L(P)]$ in $K(TX)$ by interpreting it as a map $\pi^*E \longrightarrow \pi^*F$ which is an isomorphism away from a compact set, thereby yielding a compactly supported complex over TX. We thus define the topological index of P to be $t-\mathrm{ind}([\sigma_L(P)])$. We can now state the Atiyah-Singer index theorem.

Theorem 8.1. *Let X be a compact smooth manifold, E and F smooth vector bundles over X, and P and elliptic differential operator from E to F. Then the analytical index of P is equal to the topological index. In other words, the following commutes*

$$\begin{tikzcd} \mathrm{EllDiffOp} \arrow[r] \arrow[dr, "a-\mathrm{ind}"'] & \mathrm{EllSymb} \arrow[r] & K(TX) \arrow[dl, "t-\mathrm{ind}"] \\ & \mathbb{Z} & \end{tikzcd}$$

8.2 Riemann - Roch

We begin with the classical statement of Riemann - Roch.

Theorem 8.2. *Let X be a compact connected Riemann surface and L a line bundle over X. Then $\chi(X, L) = \chi(X, \mathcal{O}) + \deg(L)$. Here, $\mathcal{O}$ is the trivial line bundle (i.e. the structure sheaf) and χ is the holomorphic Euler characteristic $\chi(X, L) = h^0(X, L) - h^1(X, L)$.*

Note that Serre duality affords a perfect pairing $H^1(X, L) \otimes H^0(X, L^* \otimes K_X) \longrightarrow \mathbb{C}$, which in the complex analytic context can be interpreted as $\alpha \otimes \beta \mapsto \int_X \alpha \wedge \beta$. Furthermore, $\chi(X, \mathcal{O})$ is computed as $1 - g$ for g the genus of X. So we can rephrase Riemann - Roch as follows.

Theorem 8.3. *$h^0(X, L) - h^0(X, L^* \otimes K_X) = 1 - g + \deg(L)$.*

In order to define some of the terms in question, as well as explain how the Atiyah - Singer index theorem is applied in proving Riemann - Roch, we must first review some basic facts about differential calculus on complex manifolds and the resulting Hodge theory. One can refer to [Huybrechts] for the missing details. For now, we content ourselves with noticing that in the second form of Riemann - Roch stated, we have a difference of analytically defined numbers on the left hand side and we have topologically defined numbers on the right hand side.

Definition 8.1. Let X be a compact connected complex manifold and TX be its real tangent bundle.

(i) $I : TX \longrightarrow TX$ is an endomorphism so that $I^2 = -\mathrm{id}$, which is defined as expected in charts.

(ii) $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$ is the complexification of the tangent bundle.

(iii) $\mathcal{T}_X$ is the holomorphic tangent bundle and its dual is Ω_X the holomorphic cotangent bundle.

(iv) $K_X = \det(\Omega_X)$ is the canonical bundle. So for instance, if X is a Riemann surface, $K_X = \Omega_X$.

(v) $T^{1,0}X$ is the i eigenspace of I on $T_{\mathbb{C}}X$ and $T^{0,1}X$ is the $-i$ eigenspace. We remark that $T^{1,0}X$ is naturally identified with the holomorphic tangent bundle $\mathcal{T}_X$, and $T^{0,1}X$ with its conjugate.

(vi) $\mathcal{A}^k_X$ is the sheaf of sections of $\bigwedge^k (T_{\mathbb{C}}X)^*$, i.e. the complex k - forms.

(vii) $\mathcal{A}^{p,q}_X$ is the sheaf of sections of $\bigwedge^{p,q} X = \bigwedge^p (T^{1,0}X)^* \otimes \bigwedge^q (T^{0,1}X)^*$. These look locally like $f dz_{i_1} \dots dz_{i_p} \overline{dz_{j_q}} \dots \overline{dz_{j_q}}$. As such, we identity $\mathcal{A}^{p,0}_X$ with the sheaf of sections of the p^{th} exterior power of the cotangent bundle Ω^p_X, i.e. the space of holomorphic p - forms.

With these definitions, we can state the essential notions of Hodge theory used in our coming application to Riemann - Roch. The foundational result in Hodge theory is the bidegree decomposition.

Proposition 8.1. *(i) There is a bidegree decomposition* $\mathcal{A}^k_X = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_X$.

(ii) The exterior derivative $d : \mathcal{A}^k_X \longrightarrow \mathcal{A}^{k+1}_X$ *splits into a sum* $\partial + \overline{\partial}$ *where* $\partial : \mathcal{A}^{p,q}_X \longrightarrow \mathcal{A}^{p+1,q}_X$ *and* $\overline{\partial} : \mathcal{A}^{p,q}_X \longrightarrow \mathcal{A}^{p,q+1}_X$. *Locally, these are given by the formulae*

$$\partial(f dz_I \overline{dz_J}) = \sum_\ell \frac{\partial f}{\partial z_\ell} dz_\ell dz_I \overline{dz_J}$$
$$\overline{\partial}(f dz_I \overline{dz_J}) = \sum_\ell \frac{\partial f}{\partial \overline{z_\ell}} \overline{dz_\ell} dz_I \overline{dz_J}$$

(iii) A function on X, *i.e. an element of* $\mathcal{A}^{0,0}_X$, *is holomorphic if and only if* $\overline{\partial} f = 0$.

Part (iii) here is just a rephrasing of the Cauchy - Riemann equations. With this, we can refine the singular cohomology of X, a topological invariant, with the analytic data of its holomorphic structure.

Definition 8.2. The Dolbeault cohomology of X, $H^{p,q}(X)$, is the q^{th} cohomology of the Dolbeault complex $(\mathcal{A}^{p,\bullet}_X, \overline{\partial})$. Its dimension is denote $h^{p,q}$, and these are called the Hodge numbers of X.

Proposition 8.2. *The singular cohomology* $H^k(X, \mathbb{C})$ *has a bidegree decomposition, induced from the above, as* $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$. *Furthermore, the Dolbeault cohomology* $H^{p,q}(X)$ *can be computed as the sheaf cohomology* $H^q(X, \Omega^p_X)$.

Proof. For the computation of sheaf cohomology, note that the Dolbeault complex is an acyclic resolution of $\mathcal{A}^{p,0}_X = \Omega^p_X$. □

Recall that in Riemann - Roch, we are interested in the cohomology of line bundles over a compact Riemann surface. As such, we would like to extend the bidegree notion to general vector bundles, in such a way that the $E = \mathcal{O}$ case recovers the above.

Definition 8.3. For a complex vector bundle $E \longrightarrow X$, we let $\mathcal{A}^{p,q}(E)$ denote the sheaf of sections of $E \otimes \bigwedge^{p,q} X$.

In particular, $\mathcal{A}^{p,q}_X = \mathcal{A}^{p,q}(\mathcal{O})$.

Proposition 8.3. *Let* $E \longrightarrow X$ *be a holomorphic vector bundle. There is a differential* $\overline{\partial}_E : \mathcal{A}^{p,q}(E) \longrightarrow \mathcal{A}^{p,q+1}(E)$ *satisfying the Leibniz rule* $\overline{\partial}_E(f\omega) = \overline{\partial} f \wedge \omega + f \overline{\partial}_E \omega$.

Via this, we form a Dolbeault complex for holomorphic vector bundles $(\mathcal{A}^{p,\bullet}(E), \overline{\partial}_E)$ whose q^{th} cohomology we denote, as above, as $H^{p,q}(X, E)$.

Proposition 8.4. *$H^{p,q}(X, E) = H^q(X, \Omega_X^p \otimes E)$.*

We now restrict our attention to Hermitian complex manifolds X and holomorphic vector bundles $E \longrightarrow X$ with a Hermitian structure h. This obstructs only the naturality of our constructions, as all complex vector bundles admit Hermitian structures but there is no canonical one. Doing so affords us a Hodge $*$ operator on X, from which we can define a Hodge $*$ operator on E. Indeed, viewing h as an isomorphism $E \xrightarrow{\sim} E^*$ we have

$$\overline{*}_E : \bigwedge^{p,q} X \otimes E \longrightarrow \bigwedge^{n-p,n-q} X \otimes E^*$$

where $n = \dim X$. This is given by $\phi \otimes s \mapsto *(\overline{\phi}) \otimes h(s)$, where $*$ is the Hodge $*$ on X.

With this, we may define an inner product on $\mathcal{A}^{p,q}(E)$ via

$$(\alpha, \beta) = \int_X \alpha \wedge \overline{*}_E \beta$$

where $\wedge$ is the usual wedge product on the form part of α, β and is the evaluation $E \otimes E^* \longrightarrow \mathbb{C}$ on the E part of α, β.

Proposition 8.5. *The map $\overline{\partial}_E^* : \mathcal{A}^{p,q}(E) \longrightarrow \mathcal{A}^{p,q-1}(E)$ defined by $\overline{\partial}_E^* = -\overline{*}_E \overline{\partial}_E \overline{*}_E$ is adjoint to the map $\overline{\partial}_E$ in this inner product.*

With this, we can define the Laplace operator $\Delta_E = \overline{\partial}_E \overline{\partial}_E^* + \overline{\partial}_E^* \overline{\partial}_E$ and develop the notion of a harmonic (p, q) form on E as an element $\alpha \in \mathcal{A}^{p,q}(E)$ so that $\Delta_E(\alpha) = 0$. We let $\mathcal{H}^{p,q}(E)$ be the space of harmonic forms in $\mathcal{A}^{p,q}(E)$. This afford us the following critical result of Hodge theory.

Theorem 8.4 (The Hodge decomposition)**.** *We have an orthogonal direct sum decomposition*

$$\mathcal{A}^{p,q}(E) = \overline{\partial}_E \mathcal{A}^{p,q-1}(E) \oplus \mathcal{H}^{p,q}(E) \oplus \overline{\partial}_E^* \mathcal{A}^{p,q+1}(E)$$

Corollary 8.4.1. *The map $\mathcal{H}^{p,q}(E) \longrightarrow H^{p,q}(E)$ is an isomorphism. That is, harmonic forms are a system of representatives of Dolbeault cohomology classes.*

Now, we can proceed return to Riemann - Roch. Take indeed a compact connected Riemann surface X and a line bundle L over X. Equip both with Hermitian metrics. Consider the differential operator

$$\overline{\partial}_L : \mathcal{A}^{0,0}(L) \longrightarrow \mathcal{A}^{0,1}(L)$$

We compute, via our above theory, its kernel and cokernel. First off, elements of $\mathcal{A}^{0,0}(L)$ are smooth sections of L. On a trivialization, these sections are smooth functions defined locally on X and $\overline{\partial}_L$ is just $\overline{\partial}$. As holomorphicity is a local property, this shows that $\ker(\overline{\partial}_L)$ consists of holomorphic sections of L. That is, $\ker(\overline{\partial}_L) = H^0(X, L)$.

Now, for its cokernel. We have $\text{cok}(\overline{\partial}_L) = \ker(\overline{\partial}_L^*)$. Here, $\overline{\partial}_L^* : \mathcal{A}^{0,1}(L) \longrightarrow \mathcal{A}^{0,0}(L)$. The kernel can be readily computed by the Hodge decomposition

$$\mathcal{A}^{0,1}(L) = \overline{\partial}_L \mathcal{A}^{0,0}(L) \oplus \mathcal{H}^{0,1}(L)$$

Note that a form is harmonic if and only if it $\overline{\partial}_L$ and $\overline{\partial}_L^*$ closed. Hence, the kernel of $\overline{\partial}_L^*$ on $\mathcal{A}^{0,1}(L)$ is precisely $\mathcal{H}^{0,1}(L)$. By the above theory, we have that $\mathcal{H}^{0,1}(L) \cong H^{0,1}(L) = H^1(X, L)$.

In summary, we have shown that $\ker(\overline{\partial}_L) = H^0(X, L)$ and $\mathrm{cok}(\overline{\partial} L) = H^1(X, L)$. As such, the index of $\overline{\partial}_L$ is precisely the Euler characteristic $\chi(X, L)$ appearing in Riemann - Roch. Hence, we have via the Atiyah - Singer index theorem

$$\chi(X, L) = t - \mathrm{ind}(\overline{\partial}_L)$$

In this context, the computation of the topological index is phrased best via cohomology and Chern classes. Rather than setting up this whole theory here, we refer the reader to [ASIII]. One sees then that

$$t - \mathrm{ind}(\overline{\partial}_L) = ch(L)Td(X)[X]$$

where ch is the Chern character, Td is the Todd class, and $[X]$ is the fundamental class of X. Note that this product of mixed cohomology classes $ch(L)Td(X)$ is itself a mixed cohomology class, so we mean here to take its top degree component. The pairing with the fundamental class can be thought of as integration

$$ch(L)Td(X)[X] = \int_X ch(X)Td(X)$$

Here, we have that the top degree component of $ch(L)Td(X)$ is $c_1(L) + \frac{1}{2}c_1(TX)$. Furthermore, $\int_X c_1(L) = \deg(L)$ and $\int_X c_1(TX) = 2 - 2g$. As such, we have shown precisely the Riemann - Roch theorem

$$\chi(X, L) = \deg(L) + 1 - g$$

by computing the left side as an analytical index, the right side as a topological index, and applying the Atiyah - Singer index theorem.

Let's note as well that this same method generalizes to proving the Hirzebruch - Riemann - Roch theorem, though we will not give many details. Indeed, consider in the case of a general holomorphic vector bundle $E \longrightarrow X$ over a compact connected complex manifold the following differential operator

$$\overline{\partial}_E + \overline{\partial}_E^* : \bigoplus_{q \text{ even}} \mathcal{A}^{0,q}(E) \longrightarrow \bigoplus_{q \text{ odd}} \mathcal{A}^{0,q}(E)$$

and compute that the analytical index is $\chi(X, E)$ and that the topological index is $\int_X ch(E)Td(X)$ to conclude, via the Atiyah - Singer index theorem, that

$$\chi(X, E) = \int_X ch(E)Td(X)$$

which is precisely the Hirzebruch - Riemann - Roch theorem.

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