Examples of adic spaces, analytic points, and goodies

First, some useful technical remarks:

i. Rational subsets form a base of \( \text{Spa}(A, A^+) \)

ii. Rationality of a subset of \( \text{Spa}(A(1/s), A^+(1/s)) \subseteq \text{Spa}(A, A^+) \)

can be tested on \( A(1/s) \) or \( A \)

These are like an "affine communication lemma" for adic spaces.

iii. \( \text{Spa}(A, A^+) \) is spectral, i.e., homeo to the Zariski spectrum of a ring

§1. Example

Notation. - \( \text{Spa}(A) = \text{Spa}(A, A^0) \)

- Let \( X \) be an adic space. Its contravariant functor is denoted \( X(-) \).

\[ Y \longrightarrow \text{Spa}(A(1/s), A^+(1/s)) \rightarrow \text{Spa}(A, A^+) \]

Remark. Fiber products don't necessarily exist in Adic Spaces, so when these are written, representability must be proven.
i. $X = \text{Spa}(\mathbb{Z}) (= \text{Spa}(\mathbb{Z}, \mathbb{Z}))$

This is final in $\text{Adic Space}$

pf. Sheafy Complete Huber pairs $\rightarrow$ $\text{Adic Space}$

$X$ is subfinal in $\mathbb{V}$.

Now, let $Y$ be adic and write $Y = \bigcup_i U_i$; $U_i$ affinoid $\mathbb{A}_k$. Then

\[ X(Y) = \bigcup_{Y,X} \bigcup_i U_i \cap X \]

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\[ = \lim_{\leftarrow i} \bigcup_{Y,X} U_i \cap X \]

\[ = \lim_{\leftarrow i} * \]

\[ = * \]

ii. $X = \text{Spa}(\mathbb{Z}[T]) (= \text{Spa}(\mathbb{Z}[T], \mathbb{Z}[T]))$

like the affine line but all functions are bounded, so we call it the adic closed unit disk.

$(\text{qim})$. $X(Y) = \partial^+ Y$

pf. A map $(\mathbb{Z}[T], \mathbb{Z}[T]) \rightarrow (A, A^*)$ is a choice of element of $A^*$; Now take, using that $\partial^+ Y$ is a sheaf.
A map \((\mathbb{Z}[T], \mathbb{Z}) \rightarrow (A, A^*)\) is an element of \(X\).

Proceeding as before, \(X(Y) = \delta_Y(Y)\).

We can lose change for previous examples to monomorphisms, for field \(K\) rather than \(\mathbb{Z}\).

iv. \(X = \text{Spec } K\). A map \((K, K^0) \rightarrow (A, A^*)\) is a compatible \(K\)-algebra and \(K^0\)-algebra structure on \(A, A^*\) resp.

so \(X(Y)\) consists of \(\delta_Y^0\) ways to make \(\delta_Y, \delta_Y^0\) into a pair of sheaves of \((K, K^0)\) -algebras.

v. Consider \(X = \text{Spec } \mathbb{Z}[T] \times \text{Spec } K\) as a functor. We claim it is represented by \(\text{Spec } K[T]\).

View \(X\) as a functor \(\text{Adr} \text{ Space} / K \rightarrow \text{Set}\), so that \(X(Y) = \delta_Y(Y)\) by UD. So in fact this is in \(K^0\text{-Alg}\),

\[\text{Rmk } (K[T], 0) \cong (K, 0)\).

Let \((A, A^*)\) be a \((K, K^0)\) -algebra (and sheafy, complete, Hayler).

A map \((K[T], K^0[T]) \rightarrow (A, A^*)\) over \(K, \text{OK is}\)

a choice of \(T\) where \(T\) is sent, say to \(aA\). Then for \(f = \sum i_n T^n \in K^0[T]\), \(f(u) = \sum i_n T^n u^n\) must converge.
Pick a $1 \in \text{Spec}(A, \mathfrak{a})$. Then by the ultrametric inequality

$$|\lambda|_n \to 0$$

$s_n \to 0$ so we need to ensure $(a^n)$ doesn’t explode, i.e., $|a| \leq 1$.

So $\text{Spec} K < \mathfrak{a} >$ represents $X$.

vi. $X = \text{Spec}(\mathbb{Z}[T], \mathbb{Z}) \times \text{Spec} K$.

Then $X(Y) = \Theta_X(Y)$, for $Y$ an adele space $\mathcal{A}/K$.

Let $\mathcal{A}/K$ be a pseudouniformizer, and $(A, A^\times)$ a sheafy complete Huber pair $(\mathcal{A}, \mathcal{O}_K)$.

Then $A = \bigcup_{n \geq 1} A^+_n$.

$(A, A^\times) \to A^+_n$ is, as before, represented by

$$\text{Spec} K < \mathcal{O}_n T >$$

Thus, $X(Y) = \bigcup_{n \geq 1} (\text{Spec} K < \mathcal{O}_n T >) (Y)$

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columns in $A^\times$ are principal, i.e. representable, are compact.

So $X$ is represented by $\bigcup_{n \geq 1} \text{Spec} K < \mathcal{O}_n T >$. 
Vii. $D = \text{Sp} \mathbb{Z} [\mathbb{C}[T]]$

A map $(\mathbb{Z} [\mathbb{C}[T]], \mathbb{Z} [\mathbb{C}[T]]) \rightarrow (A, A^+)$ is a choice of element $a \in A^+$ but since the $\mathbb{Z}$-coefficients have no "0" hypothesis, we must have $|a| < 1$ to ensure convergence.

Hence, $D$ is the adic open unit disk.

Viii. $D_{/\mathbb{R}} = D \times \text{Sp} \mathbb{K}$, represented as before by

$$U \text{Sp} \mathbb{K} \left< T, w^{-1} T^n \right>$$

ix. $D^* = \text{Sp} \mathbb{Z} [\mathbb{C}[T])]$ is the adic punctured open unit disk.

Now, $T \mapsto \text{unit in } A$

$\kappa. \quad D_{/\mathbb{R}}^* = D^* \times \text{Sp} \mathbb{K}$

$$= D_{/\mathbb{R}}^* - \frac{1}{2} \mathbb{T} = 0^2$$

xi. Let $S$ be profinite, $A = \text{Top} (S, \mathbb{Z})$, a discrete ring.

$\Sigma = \text{Sp} A \text{ represent } \bigvee \rightarrow \text{Hom} (1 \mathbb{Y}, S)$

and $|\Sigma| = S \times (\text{Sp} \mathbb{Z})$.

xii. $S_K = S \times \text{Sp} \mathbb{K}$, $|S_K| = S$. 
§2 Analytic point

We start this discussion as another example.

Let $A = \mathbb{Z}_p[[T]]$ with the $(p,T)$-adic topology. We seek Spec $A$.

First, Spec $\mathbb{Z}_p$ has 2 points, Special and General.

The same holds for $\mathrm{Spec} \mathbb{F}_p[[T]]$.

Recall (by universal prop)

\[
\mathrm{Spec} A = \begin{array}{c}
\text{irreducible distinguished} \\
\text{polynomials in } A
\end{array}
\]

Only $(0,T)$ is open in $A$, and it has a unique valuation.

\[
X|_{\mathbb{F}_p}, A \rightarrow \mathbb{F}_p \rightarrow \{0,1\}.
\]
Def. Let \((A, A^+)\) be a Huber pair.

\(x \in \text{Spur}(A, A^+)\) is non-qualistic if \(\text{Re}(1 \cdot x) \in \text{op}_1\)

and qualistic otherwise.

Let \((A_0, I)\) be a couple of definition for \(A\).

Let \(x \in \text{Spur}(A, A^+)\) qualistic. Then \(I \nmid \text{Re}(1 \cdot x)\), i.e.,
\[ \exists f \in \text{Spur}(A, A^+) \text{, } (f(x)) \neq 0 \text{, call this } f \in \mathcal{P}. \]

As \(f \in \mathcal{P}\), \(\lim_{n \to \infty} \mathcal{P} = \text{max}\).

Using this, we can show that \(\exists ! f \to \mathbb{R}^{\geq 0} \text{ s.t. } \gamma \to \frac{1}{2}. \)

As such, \(K(x)\) admits a nonarchimedean norm valued in \(\mathbb{R}^{\geq 0}\).

Def. Let \(x \in \text{Spur}(A, A^+)\), \(K(x) = \bigcup_{\text{qualistic, } x \text{, } \gamma \text{, } \text{max}} \mathcal{P} \to \mathbb{R}^{\geq 0}\)

With this, we can identify points of \(\text{Spur}(A, A^+)\) with rings of fields, qft the Zariski spectrum.
Def. An affinoid field is a Haber pair $(K, K^+)$ with $K$ a nonzero discrete and $K^+$ an open, bounded valuation ring.

For $x$, take $K(x)$. This has a valuation $v$, hence a valuation ring $K(x)^+$. $(K(x), K(x)^+)$ is an affinoid.

Prop. $\text{Spa}(A, A^+) \cong \left\{ (A, A^+) \rightarrow (K, K^+) \mid (K, K^+) \text{ affinoid and } A \rightarrow K \text{ has } \text{dvr} \right\}$

Further, if $x \in \text{Spa}(A, A)$ analytic, corresponding to $(A, A^+) \rightarrow (K(x), K(x)^+)$, then $x$ is generalized to $y$ corresponding to $(A, A^+) \rightarrow (K(y), K(y)^+)$ with $K(x) \subseteq K(y)$ and $K(y)^+ \subseteq K(x)^+$

so generalization of $x$'s are to set of length rank($x$).
Back to $\chi = \text{Spa} A$, let $Y = X - \{ x \in F_p \}$. 

Prop. $\exists \alpha \in \text{cont } Y \rightarrow \mathbb{C}o, \text{alg } c.$

\[
\alpha(y) = u \leq \lim_{\nu} b \left( \frac{m}{n} \right) \frac{\mid \tau(x) \mid}{\mid \rho(x) \mid^{m/n}}
\]

\[\quad \text{and } a \left( \frac{m}{n} \right) < u, \quad \mid \tau(x) \mid \leq \mid \rho(x) \mid^{m/n}\]

Let $Y(0, \infty) = \text{int} \left( \alpha^{-1} [0, \infty) \right)$, a Fréchet manifold $\mathcal{F}$.

Note $\alpha(x) = 0 \implies 1_{\tau(x)} = 0$, so this is smooth and is 

thus the generic fiber $\text{Spa} \Lambda \rightarrow \text{Spa} \mathbb{Z}_p$, $Y(0, \infty) \rightarrow (0, \infty)$ 

so this is not compact. Hence, non-quotient.
Naively, this seems trivial, but isn't it?

Yes, $\text{Seq} \left( \Lambda \left[ \text{Ev}_n \right], \Lambda \right)$, but how to tensorize $\Lambda [1/p]$?

If $(\mathbb{Z}/1/p)$-adic, $p^i T^n \to 0$ but never exact.

So this is not Huber.

If $m \geq 2$, $\Lambda \to \Lambda [1/p]$ not cts, so $T^n \to 0$ $(\mathbb{Z}/1/p)$-additive.

It's not affinoid, can why $\text{Ev}(1/H^2)$ be added?

Let $x \in Y_{(0, \infty)}$, $(T^n(x)) \to 0$ and $|P(x)| \to$

$\exists n > 20$ s.t. $(T^n(x)) \leq |P(x)|$.

Thus, $x \in \text{Sp} \left( \Lambda_{(1, T^n/1/p)} \right)$, which is not cts, and $\Lambda_{(1, T^n/1/p)}$ is open.

Note $Y_{(0, \infty)} = \text{Sp} \left( \Lambda_{(1, T^n/1/p)} \right)$, which covers $Y_{(0, \infty)}$.

Essentially, $\sum_{i \geq 1} = \prod_{i \geq 1} \prod_{i \leq 1} |T_i| \leq |P(1)|^{1/3}$.

Finally, what about $Y_{(1/P)}(17)$? Let $U = Y_{(0, 13)}$.

This is not-affinoid, so $\prod_{i \geq 1} |P_i(x)| \leq (T^n(x)) \to 0$, $\Theta : U \mapsto \Lambda [1/17]$ complete torsion topology on $\Lambda [1/17]$. This is Tate (visibly) not unramified on fields, hence non-trivial identity.
Recall, a Huber ring \( A \) is called quasifl \( \text{ if the ideal of topologically nilpotent elements of } A \) is the unit ideal.

Prop. Let \( (A, A^+) \) be a complete Huber pair.

i. A quasifl \( \implies \) all points of \( \text{ Spec}(A, A^+) \) are quasifl.

ii. \( \text{ Spec}(A, A^+) \) is quasifl \( \iff \exists r \in U \subseteq \text{ Spec}(A, A^+) \text{ real } \iff \mathcal{O}_x(U) \text{ is Tate.} \)

Def. Let \( I \) be this ideal, then quasifl \( \iff \in U(2) \)

ii. If \( A \text{ is quasifl, } \exists f \in I \text{ s.t. } |f(x)| \leq 0. \)

\( g \in A \) \( \text{ if } |g(x)| \leq |f(x)| \text{ for all } x \in \mathbb{R} \), and \( x \) is rational, then \( x \in I \).

Let \( I^+ = (g_{<1}, g_{\geq 1}) \). Then \( f \) is a unit on the rational subset \( U \left( \frac{g_{\geq 1} - g_{<1}}{t} \right) \) and is topologically nilpotent.

Prop. Let \( A \) be an quasifl Huber ring. \( M, N \) complete Banach \( A \)-modules, and \( M \twoheadrightarrow N \) flat, surj. Then when \( i \) is then open.
we may restrict to a class of phenomena resembling Analitiy.

Def. $f: A \rightarrow B$ of ring rings is called adic if its sends $A_0$ to $B_0$ (units of $A_0$), and for $I \subseteq A_0$ ideal on $A_0$, $f(I)B_0$ is an ideal of $B_0$.

For $T \in T_A$, then $f$ adic.

Indeed, Tate rings admit ideals of depth $p \geq 0$

a power of a pseudo-uniformizer.

In $T$, where pseudo-unit of $T$, $f(A)$ is a pseudo-unit

of $B$ so then $f$ an ideal of $A$ and $B$ by some $f(A)$.

Prop. $(A, A^+ \rightarrow (B, B^+)$ of commutative rings is

adic if $\text{Spec}(B, B^+) \rightarrow \text{Spec}(A, A^+)$ sends

qualitive points to analytic points.
Prop. i. \((A, A^+) \rightarrow (B, B^+)\) adic. Then on

\begin{align*}
\text{some \ rat'\!\!'s subets pull back to rat'\!\!'s substes.}
\end{align*}

If we complet (D,D^+), this is a pullback in

\begin{align*}
\text{Completes Huber rings.}
\end{align*}

This is of course a pullback of affinoid adic

spaces, but also of adic spaces. However,

let T be aff. X X \rightarrow a pullback of affinoid adic

spaces

\begin{align*}
V(T, X_{x,y}) & = \lim_{\rightarrow} V(\nu_i, X_{x,y}) = \lim_{\rightarrow} \nu(\nu_i) \times_{\nu(\nu_i)} X_{x,y} \\
& \subseteq X(T) \times_{\nu(\nu_i)} X(y) \\
& \subseteq X(T) \times_{\nu(\nu_i)} X(y).
\end{align*}
Indeed, we claim \((\mathbb{Z}_p, \mathbb{Z}_p) \rightarrow (\mathbb{Q}_p, \mathbb{Z}_p)\)
\[
\downarrow
\]
\[
(1, 1)
\]

has no pushout. So \((0, 0^+)\) was a pushout.

Then have \((0, 0^+) \rightarrow (\mathbb{Q}_p < T, \mathbb{Z}_p < T^* / \mathbb{Q}_p)\),

but \(T < 1\) is topologically nilpotent, so we have a map \(\lambda \rightarrow 0\), \(\mathbb{Z}_p \rightarrow \mathbb{Z}_p^0\) and

hence is eventually in \(0^+\). But if \(\mathbb{Z}_p^0 \rightarrow \mathbb{Z}_p\), we

cannot map to \(\mathbb{Z}_p < T, \mathbb{Z}_p^* / \mathbb{Q}_p\),

In other words, \(\text{Sh}_{A \times B} \text{Sh}_{B} (\mathbb{Q}_p)\) is not representable

by an affine \(\mathbb{Q}_p\) space.
§3. hoodie

i. Recall a uniform is $A$ bounded.

Thm (Birkhoff). $A \rightarrow \prod_{K(x) \in k_{Stv}(A, A^t)}$ homoeote its image for $A$ uniform

(Or. $A \rightarrow \partial x(u)$, as the latter embeds into the product

and isiz2

ii. Def. $(A, A^t)$ a complete Huber pair is strongly uniform

if $\partial x(u)$ is uniform for all real subrings.

Then, There are Ssheafy

iii. Let $(A, A^t)$ be a Ssheaf and pseudo Huber pair. Then $H^i (\gamma, \partial x) = 0$ for $i > 0$.

iv. Let $(A, A^t)$ be Ssheaf pseudo Huber, Then $\Sigma_{A \rightarrow 0} \rightarrow$ locally finite for $\partial x$-Mod.