Examples of adic spaces, analytic points, and goodies

First, some useful technical remarks:

i. Rational subsets form a base of $\mathrm{Spa}(A, A^+)$

ii. Rationality of a subset of $\mathrm{Spa}(A\langle T/s\rangle, A^+[T/s]) \subseteq \mathrm{Spa}(A, A^+)$ can be tested on $A\langle T/s\rangle$ or A

These are like an "affine communication lemma" for adic spaces.

iii. $\mathrm{Spa}(A, A^+)$ is spectral i.e. homeo to the Zariski spectrum of a ring

§1. Examples

Notation. - $\mathrm{Spa}(A) = \mathrm{Spa}(A, A^\circ)$

- Let X be an adic space. Its contravariant hom functor is denoted $X(-)$: $\text{Adic Spaces}^{op} \longrightarrow \text{Set}$

$$Y \longmapsto V(Y, X)$$

Rmk. Fiber products don't necessarily exist in Adic Spaces, so when these are written, representability must be proven.

i. $X = \mathrm{Spa}(\mathbb{Z})\ \left(= \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})\right)$

This is final in Adic Spaces

Pf. $$\text{Sheafy Complete Huber Pairs}^{op} \xhookrightarrow[\text{fully faithful}]{} \text{Adic Spaces}$$

$$\downarrow$$

X is surely final in $\mathcal{V}$.

Now, let Y be adic and write $Y = \bigcup U_i$, U_i affinoid adic. Then

$$\begin{aligned} X(Y) &= \mathcal{V}(Y, X) \\ &= \mathcal{V}(\textstyle\bigcup U_i, X) \\ &= \varprojlim_i \mathcal{V}(U_i, X) \\ &= \varprojlim_i * \\ &= * \end{aligned}$$

ii. $X = \mathrm{Spa}\, \mathbb{Z}[T]\ \left(= \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}[T])\right)$

like the affine line but all functions are bounded, so we call it the <u>adic closed unit disk</u>.

Claim: $X(Y) = \mathcal{O}_Y^+(Y)$

Pf. A map $(\mathbb{Z}[T], \mathbb{Z}[T]) \longrightarrow (A, A^+)$ is a choice of element of A^+. Now glue, using that $\mathcal{O}_Y^+$ is a sheaf.

iii. $X = \mathrm{Spa}(\mathbb{Z}[T], \underbrace{\mathbb{Z}}_{\text{no boundedness condition at all}})$, the adic affine line

A map $(\mathbb{Z}[T], \mathbb{Z}) \to (A, A^+)$ is an element of A

Proceeding as before, $X(Y) = \mathcal{O}_Y(Y)$

We can base change the previous examples to nonarchimedean base fields K rather than $\mathbb{Z}$

iv. $X = \mathrm{Spa}\, K$. A map $(K, K^o) \to (A, A^+)$ is a compatible K-alg and K^o-alg structure on A, A^+ resp.

So $X(Y)$ consists of the ways to make $(\mathcal{O}_Y, \mathcal{O}_Y^+)$ into a pair of sheaves of (K, K^o)-algebras.

v. Consider $X = \mathrm{Spa}\, \mathbb{Z}[T] \times \mathrm{Spa}\, K$ as a functor. We claim it is represented by $\mathrm{Spa}\, K\langle T\rangle$,

View X as a functor $\mathrm{Adic\ Spaces}/K^{op} \to \mathrm{Set}$, so that $X(Y) = \mathcal{O}_Y^+(Y)$ by UP. So in fact, this is in K^o-Alg,

Rmk. $(K\langle T\rangle)^o = \mathcal{O}_K\langle T\rangle$

Let (A, A^+) be a $(K, \mathcal{O}_K)$ algebra (and sheafy, complete, Huber).
A map $(K\langle T\rangle, \mathcal{O}_K\langle T\rangle) \to (A, A^+)$ over $K, \mathcal{O}_K$ is a choice of where T is sent, say to $a \in A$. Then for $f = \sum \lambda_n T^n \in K\langle T\rangle$, $f(a) = \sum \lambda_n T^n$ must converge.

Pick a $|\cdot| \in \operatorname{Spa}(A, A^+)$. Then by the ultrametric inequality

$$|\lambda_n a^n| \longrightarrow 0$$

$|\lambda_n| \longrightarrow 0$ so we need to assure $|a|^n$ doesn't explode,

i.e. $|a| \leq 1$.

So $\operatorname{Spa} K\langle T\rangle$ represents X

vi. $X = \operatorname{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \operatorname{Spa} K$

Then $X(Y) = \mathcal{O}_Y(Y)$, for Y an adic space $/K$.

Let $\varpi \in K$ be a pseudouniformizer, and (A, A^+)

a sheafy complete Huber pair $/(K, \mathcal{O}_K)$.

Then $A = \bigcup_{n \geq 1} \varpi^{-n} A^+$

$(A, A^+) \longmapsto \varpi^{-n} A^+$ is, as before, represented by

$\operatorname{Spa} K\langle \varpi^n T\rangle \quad \left(|\varpi^n T| \leq 1 \iff |T| \leq |\varpi|^{-n} \right)$

Thus, $X(Y) = \bigcup_{n \geq 1} (\operatorname{Spa} K\langle \varpi^n T\rangle)(Y)$

$= \left(\bigcup_{n \geq 1} \operatorname{Spa} K\langle \varpi^n T\rangle \right)(Y)$ colimits in Psh are pointwise, i.e. representables are compact

So X is represented by $\bigcup_{n \geq 1} \operatorname{Spa} K\langle \varpi^n T\rangle$.

vii. $\mathbb{D} = \mathrm{Spa}\, \mathbb{Z}[[T]]$

A map $(\mathbb{Z}[[T]], \mathbb{Z}[[T]]) \longrightarrow (A, A^+)$ is a choice of element $a \in A^+$, but since the $\mathbb{Z}$ coeffs have no "$\to 0$" hypothesis, we must have $|a| < 1$ $\forall$ $|\cdot|$ to ensure convergence.

Hence, $\mathbb{D}$ is the <u>adic open unit disk</u>

viii. $\mathbb{D}_K = \mathbb{D} \times \mathrm{Spa}\, K$, represented as before by

$$\bigcup_{n \geq 1} \mathrm{Spa}\, K\langle T, \varpi^{-1} T^n \rangle$$

ix. $\mathbb{D}^* = \mathrm{Spa}\, \mathbb{Z}((T))$ the <u>adic punctured open unit disk</u>

Now, $T \longmapsto$ unit in A^+

x. $\mathbb{D}^*_K = \mathbb{D}^* \times \mathrm{Spa}\, K$

$= \mathbb{D}_K - \{T = 0\}$

xi. Let S be profinite, $A = \mathrm{Top}(S, \mathbb{Z})$, a discrete ring.

$\underline{S} = \mathrm{Spa}\, A$ represents $Y \longmapsto \mathrm{Hom}(|Y|, S)$

and $|\underline{S}| = S \times |\mathrm{Spa}\, \mathbb{Z}|$.

xii. $\underline{S}_K = \underline{S} \times \mathrm{Spa}\, K$, $|\underline{S}_K| = S$.

§2 Analytic points

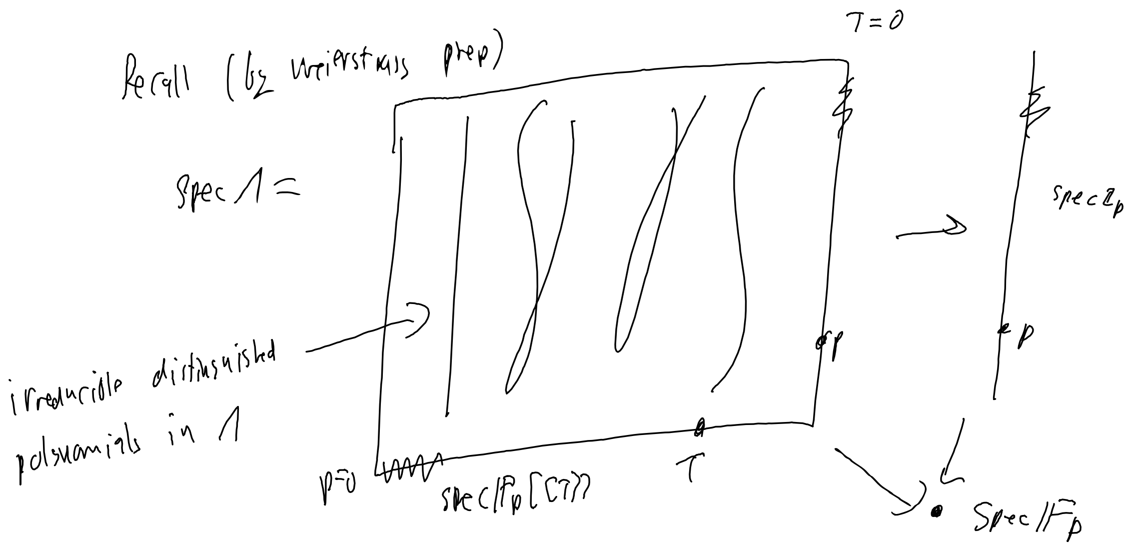
We start this discussion as another example.

Let $\Lambda = \mathbb{Z}_p[[T]]$ with the (p, T)-adic topology, we seek $\mathrm{Spa}\,\Lambda$.

First, $\mathrm{Spa}\,\mathbb{Z}_p$ has 2 points, special and generic.

The same holds for $\mathbb{F}_p[[T]]$.

Recall (by Weierstrass prep)



Only (p, T) is open in Λ, and it has a unique valuation

$$\chi_{\mathbb{F}_p} : \Lambda \longrightarrow \mathbb{F}_p \longrightarrow \{0, 1\}.$$

Def. Let (A, A^+) be a Huber pair.

$x \in \mathrm{Spa}(A, A^+)$ is non-analytic if $\ker(|\cdot|_x)$ is open

and analytic otherwise

Let (A_0, I) be a couple of definition for A.

Let $x \in \mathrm{Spa}(A, A^+)$ analytic. Then $I \not\subseteq \ker(|\cdot|_x)$, i.e.,

$\exists f \in I$ s.t. $|f(x)| \neq 0$. Call this $\gamma \in \Gamma$.

As $f \in I$, $f^n \to 0$, so $\lim\limits_{n \to \infty} \inf \gamma^n = \inf \Gamma$

Using this, we can show that $\exists! \; \Gamma \longrightarrow \mathbb{R}^{\geq 0}$ s.t. $\gamma \mapsto \frac{1}{2}$.

As such, $K(x)$ admits a nonarchimedean norm valued in $\mathbb{R}^{\geq 0}$.

Def. Let $x \in \mathrm{Spa}(A, A^+)$. $\mathcal{K}(x) = \begin{cases} \widehat{K(x)} & x \text{ analytic} \\ \underset{\text{discrete}}{\underset{\uparrow}{K(x)}} & x \text{ nonanalytic} \end{cases}$

With this, we can identify points of $\mathrm{Spa}(A, A^+)$ with maps to fields, a la the Zariski Spectrum.

Def. An affinoid field is a Huber pair (K, K^+) w/ K nonarch or discrete and K^+ an open, bounded val'n ring

e.g. Take $K(x)$. This has a val'n via x, hence a val'n ring $K(x)^+$. $(K(x), K(x)^+)$ is affinoid

Prop. $\mathrm{Spa}(A, A^+) \xleftrightarrow{\sim} \left\{ (A, A^+) \to (K, K^+) \;\middle|\; \begin{array}{l} (K, K^+) \text{ affinoid} \\ \text{and } A \to K \text{ has} \\ \text{dense image} \end{array} \right\}$

Further, if $x \in \mathrm{Spa}(A, A^+)$ analytic, corresponding to $(A, A^+) \to (K(x), K(x)^+)$, then its generalizations y correspond to $(A, A^+) \to (K(y), K(y)^+)$ with $K(x) = K(y)$ and $K(x)^+ \subseteq K(y)^+$

So $\{$generalizations of x's$\}$ are a toset of length $\mathrm{rank}(x)$.

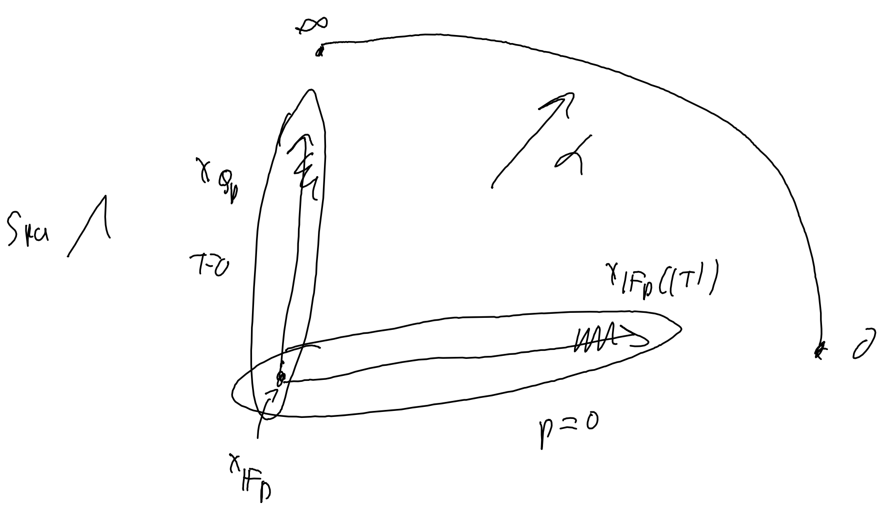
Back to $X = \mathrm{Spa}\,\Lambda$, let $\mathcal{Y} = X - \{x_{\mathbb{F}_p}\}$.

Prop. $\exists!$ cont $\mathcal{Y} \xrightarrow{\alpha} [0,\infty]$ s.t.

$\alpha(x) = r \iff \forall \frac{m}{n} > r$, $|T(x)| \geq |p(x)|^{m/n}$

and $\forall \frac{m}{n} < r$, $|T(x)| \leq |p(x)|^{m/n}$

α is onto.



Let $\mathcal{Y}_{(0,\infty]} = \mathrm{int}(\alpha^{-1}[(0,\infty]])$, or for any interval I,

Note $\alpha(x) = 0 \iff |p(x)| = 0$, so this is $\{p \neq 0\}$ and is thus the generic fiber $\mathrm{Spa}\,\Lambda \longrightarrow \mathrm{Spa}\,\mathbb{Z}_p$. $\mathcal{Y}_{(0,\infty]} \overset{\alpha}{\twoheadrightarrow} (0,\infty]$ so this isn't compact. Hence, non-affinoid.

Naively, this generic fiber would just invert p,

e.g., $\mathrm{Spa}\,(\Lambda[1/p], \Lambda)$. But how to topologize $\Lambda[1/p]$?

If (p,T)-adic, $p^{-1}T^n \longrightarrow 0$ but never enters Λ,

so this is not Huber.

If p-adic, $\Lambda \longrightarrow \Lambda[1/p]$ not cts, as

$T^n \longrightarrow 0$ (p,T)-adically but not p-adically.

It's not affinoid, but why explicitly is it adic?

Let $x \in \mathcal{Y}_{(0,\infty]}$. $|T^n(x)| \longrightarrow 0$ and $|p(x)| \neq 0$

so $\exists\, n \gg 0$ s.t. $|T^n(x)| \leq |p(x)|$.

Thus, $x \in \mathrm{Spa}\,\mathbb{Q}_p\langle T, T^n/p\rangle$ which is rational as (T^n, p) is open.

Note $\mathcal{Y}_{[1/n,\infty]} = \mathrm{Spa}\,\mathbb{Q}_p\langle T, T^n/p\rangle$, which cover $\mathcal{Y}_{(0,\infty]}$.

essentially, $\{|T|<1\} = \bigcup_{n \geq 1} \{|T| \leq |p|^{1/n}\}$.

Finally, what about $x_{\mathbb{F}_p((T))} \in \mathcal{Y}$ itself? Let $U = \mathcal{Y}_{[0,1]}$.

This is rational: $\{|p(x)| \leq |T(x)| \neq 0\}$. $\mathcal{O}_x(U)$ is $\Lambda[1/T]$ completed at the T-adic topology on $\Lambda[p/T]$. This is Tate (via T) but contains no nonarchimedean fields, hence non-rigid entirely!

Recall A Huber is called analytic if the ideal of topologically nilpotent elements of A is the unit ideal.

Tate $\Rightarrow$ analytic.

Prop. Let (A, A^+) be a complete Huber pair.

i. A analytic $\Leftrightarrow$ all points of $\mathrm{Spa}(A, A^+)$ analytic

ii. $x \in \mathrm{Spa}(A, A^+)$ analytic $\Leftrightarrow \exists\, x \in U \subseteq \mathrm{Spa}(A, A^+)$ rational s.t. $\mathcal{O}_X(U)$ is Tate

Pf. i. Let I be this ideal. x non analytic $\Leftrightarrow x \in V(I)$

ii. If x analytic, $\exists\, f \in \underset{\text{ideal of def}}{I}$ s.t. $|f(x)| \neq 0$.

$\{g \in A \mid |g(x)| \leq |f(x)|\}$ is open, so it contains some I^n.

Let $I^n = (g_1, \ldots, g_n)$. Then f is a unit on the rational subset $U\left(\frac{g_1, \ldots, g_n}{f}\right)$ and is topologically nilpotent.

Prop. Let A be an analytic Huber ring, M, N complete Banach A-modules, and $M \longrightarrow N$ cont. surj. This map is then open.

We now restrict to a class of morphisms respecting analyticity.

Def. $f: A \to B$ of Huber rings is called adic if it sends A_0 to B_0 (rings of def), and for $I \subseteq A_0$ ideal of def, $f[I]B_0$ is an ideal of def.

e.g. A Tate then f adic.

Indeed, Tate rings admit ideals of def's gen by a power of a pseudo uniformizer.

Fix ϖ is a pseudounif of A, $f(\varpi)$ is a pseudo-unif of B so there is an ideal of def'n gen by some $f(\varpi^n)$.

Prop. $(A, A^+) \to (B, B^+)$ of complete Huber pairs is adic iff $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ sends analytic points to analytic points.

Prop. i. $(A, A^+) \to (B, B^+)$ adic (i.e. $A \to B$ adic), then on Spa rat'l subsets pull back to rat'l subsets

ii. pushouts exist in Huber pairs when the legs are adic.

$$\begin{tikzcd}
(A, A^+) \arrow[r, "\text{adic}"] \arrow[d, "\text{adic}"'] & (B, B^+) \arrow[d] \\
(C, C^+) \arrow[r] & (B \otimes_A C, \overbrace{B^+ \otimes_{A^+} C^+}^{\text{int. closure}})
\end{tikzcd}$$

If we complete (D, D^+), this is a pushout in complete Huber pairs.

Rmk. This is of course a pullback of affinoid adic spaces, but also of adic spaces themselves,

$X \times_Z Y$ a pullback of affinoid adic spaces

Let T be adic, $= \bigcup U_i$ affinoid

$$\mathcal{V}(T, X \times_Z Y) = \lim_{\leftarrow} \mathcal{V}(U_i, X \times_Z Y) = \lim_{\leftarrow} X(U_i) \times_{Z(U_i)} Y(U_i)$$

$$= X(T) \times_{Z(T)} Y(T).$$

e.g. $\mathbb{Z}_p \longrightarrow A$ is <u>not</u> adic.

Indeed, we claim

$$\begin{tikzcd}
(\mathbb{Z}_p, \mathbb{Z}_p) \arrow[r, "\text{clearly adic}"] \arrow[d] & (\mathbb{Q}_p, \mathbb{Z}_p) \\
(A, A) &
\end{tikzcd}$$

has no pushout. Say (D, D^+) was a pushout. Then have $(D, D^+) \to (\mathbb{Q}_p\langle T, T^n/p\rangle, \mathbb{Z}_p\langle T, T^n/p\rangle)$, but $T \in D$ is topologically nilpotent as we have a map $A \longrightarrow D$, so $\frac{T^n}{p} \longrightarrow 0$ in D and hence is eventually in D^+. But if $\frac{T^n}{p} \in D^+$, we cannot map to $\mathbb{Z}_p\langle T, T^n/p\rangle$.

In other words, $\mathrm{Spa}\, A \times_{\mathrm{Spa}\,\mathbb{Z}_p} \mathrm{Spa}(\mathbb{Q}_p)$ is not representable by an affinoid adic space!

§3. Goodies

i. Recall A uniform if A° bounded.

Thm (Berkovich). $A \longrightarrow \prod_{x \in \mathrm{Spa}(A, A^+)} K(x)$ is

homeomorphic on image for A uniform

Cor. $A \hookrightarrow \mathcal{O}_X^{sh}(X)$, as the latter embeds into the product

ii. Def. (A, A^+) a complete analytic Huber pair is stably uniform if $\mathcal{O}_X(U)$ is uniform for all rat'l subsets U

Thm. These are sheafy

iii. Let (A, A^+) be a sheafy complete analytic Huber pair.

Then $H^i(X, \mathcal{O}_X) = 0$ $\forall i > 0$.

iv. Let (A, A^+) be sheafy analytic Huber. Then

$$\text{f.g. } A\text{-Proj} \xrightarrow{\simeq} \text{locally finite free } \mathcal{O}_X\text{-Mod}$$