1 Extension of Scalars

Let R be a commutative ring. Recall briefly the notion of an R-algebra

Definition 1.1. An R-algebra is a map $f : R \longrightarrow S$ where S is a commutative ring. This means that S is a commutative ring with the structure of an R-module via the formula

$$rs := f(r)s$$

for $r \in R$ and $s \in S$.

Since S is an R-module it may be tensored (over R) by other R-modules, which will yield for us *extension of scalars*. Before we get into that, we first define the pullback or restriction of scalars.

Definition 1.2. Let S be an R-algebra via $f : R \longrightarrow S$. Let M be an S-module. We define an R-module f^*M which is the same underlying abelian group as M equipped with the R-action

$$rm := f(r)m$$

using the existing S-module structure on M. Often, we will be lazy and suppress the f^* from the notation, and simultaneously refer to M as an S-module and as an R-module via this formula.

For example, let V be a vector space over $\mathbb{C}$. Consider the inclusion map $i : \mathbb{R} \longrightarrow \mathbb{C}$. This makes $\mathbb{C}$ into an $\mathbb{R}$-algebra, so we can restrict scalars from $\mathbb{C}$ to $\mathbb{R}$ to get a vector space i^*V over $\mathbb{R}$. This is the complex vector space V viewed as a real vector space in the usual way. Hence, we have restricted the scalars from $\mathbb{C}$ to $\mathbb{R}$. This works the same way for any field extension.

Now we go the other way, which requires the tensor product.

Definition 1.3. Let $f : R \longrightarrow S$ be an R-algebra and M be an R-module. We consider the R-module $S \otimes_R M$, sometimes called f_*M, and equip it with an S-module structure via

$$s(s' \otimes m) = (ss') \otimes m$$

To be a bit more formal, fix $s \in S$ and define the R-bilinear map $S \times M \longrightarrow S \otimes_R M$ via $(s', m) \mapsto (ss') \otimes m$. Then use the universal property of the tensor product to define the map as above.

Now there is an induced R-linear map $M \longrightarrow S \otimes_R M$ via $m \mapsto 1 \otimes m$. We claim that this is the "most efficient" way to transform M from an R-module into an S-module. Formally, we assert the following universal property.

Theorem 1.1. *Let S be an R-algebra via $f : R \longrightarrow S$ and let N be an S-module and suppose there is an R-linear map $\phi : M \longrightarrow N$. Then there is a unique S-linear map $\Phi : S \otimes_R M \longrightarrow N$ so that the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d] & N \\ S \otimes_R M \arrow[ur, dashed, "\exists!\Phi"'] & \end{tikzcd}$$

Remarks. (i) We said N was an S-module and then asserted that $\phi : M \longrightarrow N$ was R-linear. This is meant to be read as saying that N is an R-module from the restriction of scalars via $f : R \longrightarrow S$. So we could have said let $\phi : M \longrightarrow f^*N$ be R-linear, but this is cumbersome so we choose to suppress it.

(ii) This is remarkably close to the universal property of localization! This is not an accident philosophically or literally. On a philosophical level, this sort of universal property is what we expect from the "most efficient" way to perform a construction. Both universal properties are examples of an "adjunction" in category theory. On a literal level, we will show later that localization of modules is an example of extension of scalars.

Proof. We construct the map Φ. Indeed, we define it via the universal property of the tensor product by sending

$$s \otimes m \mapsto s\phi(m)$$

where, as usual, we mean to take the R-bilinear map $S \times M \longrightarrow N$ via

$$(s, m) \mapsto s\phi(m)$$

and induce the map $\Phi : S \otimes_R M \longrightarrow N$ by universal property.

Let's check that this is R-bilinear. By ϕ being a group homomorphism and distributivity of multiplication, it is surely biadditive (aka $\mathbb{Z}$-bilinear). So we check that this map respects R scalars. First of all,

$$\begin{aligned}(rs, m) &\mapsto (rs)\phi(m) \\ &= (f(r)s)\phi(m) \\ &= f(r)(s\phi(m)) \\ &= r(s\phi(m))\end{aligned}$$

The second line is by definition of the R-module structure on S via f, and the last line is by definition of the R-module structure on N via restriction of scalars along f. On the other hand, we have

$$\begin{aligned}(s, rm) &\mapsto s\phi(rm) \\ &= s(r\phi(m)) \\ &= s(f(r)\phi(m)) \\ &= (sf(r))\phi(m) \\ &= (rs)\phi(m) \\ &= r(s\phi(m))\end{aligned}$$

Hence, this map is R-bilinear so Φ is R-linear by universal property of the tensor product.

Furthermore, we see that $\Phi(1 \otimes m) = 1\phi(m) = \phi(m)$, so the diagram does indeed commute.

Now, we have constructed an R-linear map $\Phi : S \otimes_R M \longrightarrow N$ which commutes in the diagram, so we now want to prove that it is in fact S-linear. Indeed, we check this on simple tensors, which are generators, so we compute

$$\begin{aligned}\Phi(s(s' \otimes m)) &= \Phi((ss') \otimes m) \\ &= (ss')\phi(m) \\ &= s(s'\phi(m)) \\ &= s\Phi(s' \otimes m)\end{aligned}$$

so we have shown S-linearity of Φ.

Finally, we show that Φ is unique. Suppose there was an S-linear map $\Psi : S \otimes_R M \longrightarrow N$ which commuted in this diagram, i.e.

$$\Psi(1 \otimes m) = \phi(m)$$

Then for any $s \in S$, we have by S-linearity that

$$\Psi(s \otimes m) = \Psi(s(1 \otimes m)) = s\Psi(1 \otimes m) = s\phi(m) = \Phi(s \otimes m)$$

So as simple tensors generate $S \otimes_R M$, we have shown $\Psi = \Phi$, proving uniqueness. $\square$

Remark. Extension of scalars is "functorial". That is, given an R-linear map $f : M \longrightarrow N$ we have the induced map $\mathrm{id} \otimes f : S \otimes_R M \longrightarrow S \otimes_R N$ which is S-linear.

Examples. (i) Let V be a vector space over $\mathbb{R}$. As before, the inclusion $i : \mathbb{R} \longrightarrow \mathbb{C}$ makes $\mathbb{C}$ into an $\mathbb{R}$-algebra. Then $\mathbb{C} \otimes_{\mathbb{R}} V$ is the extension of scalars of V to $\mathbb{C}$. If V has basis $\{\beta_i\}_{i \in I}$ then we may write

$$V \cong \coprod_{i \in I} \mathbb{R}\beta_i$$

We have that the tensor product commutes with the coproduct, so

$$\mathbb{C} \otimes_{\mathbb{R}} V \cong \coprod_{i \in I} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}\beta_i$$

and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}\beta_i$ is a one dimensional $\mathbb{C}$ vector space with basis $1 \otimes \beta_i$. Thus, we conclude that $\mathbb{C} \otimes_{\mathbb{R}} V$ is the complex vector space with basis $\{1 \otimes \beta_i\}_{i \in I}$. Essentially, we have just replaced $\mathbb{R}$ with $\mathbb{C}$.

(ii) Suppose we have an R-module with an explicit presentation like $M = R\langle x_1, x_2 | 2x_1 = x_2 \rangle$. Formally, this means that there is an exact sequence

$$\begin{array}{ccccccc} R & \longrightarrow & R^2 & \longrightarrow & M & \longrightarrow & 0 \\ 1 & \longmapsto & (2,-1) & & & & \\ & & e_i & \longmapsto & x_i & & \end{array}$$

Now, let $R \longrightarrow S$ be an R-algebra. We tensor this sequence by S to get a presentation of $S \otimes_R M$. Indeed, by right exactness of the tensor product, the following is an exact sequence of S-modules.

$$S \otimes_R R \longrightarrow S \otimes_R R^2 \longrightarrow S \otimes_R M \longrightarrow 0$$

We have isomorphisms $S \otimes_R R \longrightarrow S$ and $S \otimes_R R^2 \longrightarrow S^2$ via $s \otimes r \mapsto sr$ and $s \otimes (a, b) \mapsto (sa, sb)$. We can thus form

$$\begin{tikzcd} S \otimes_R R \arrow[r] \arrow[d, "\sim"] & S \otimes_R R^2 \arrow[r] \arrow[d, "\sim"] & S \otimes_R M \arrow[r] & 0 \\ S \arrow[r] & S^2 \arrow[ur] & & \end{tikzcd}$$

We compute the image of 1 in $S \longrightarrow S^2$ via commutativity of this square. Indeed,

$$\begin{tikzcd} 1 \otimes 1 \arrow[r, mapsto] & 1 \otimes (2,-1) \arrow[d, mapsto] \\ 1 \arrow[u, mapsto] & (2,-1) \end{tikzcd}$$

so this is given by $1 \mapsto (2, -1)$. Similarly, $S^2 \longrightarrow S \otimes_R M$ is given by $e_i \mapsto 1 \otimes x_i$. Write $y_i = 1 \otimes x_i$.

We conclude that we have the presentation

$$S \otimes_R M = S\langle y_1, y_2 | 2y_1 = y_2 \rangle$$

Thus, when given a presentation of a module, we extend scalars by just replacing R with S.

2 Localization

As discussed previously, there is a close relationship between localization and extension of scalars, which we now explore. For one, both localization and extension of scalars are ways to turn R-modules into $S^{-1}R$-modules. Let's first recall localization of modules.

Let R be a commutative ring with a multiplicative subset S. For an R-module M we define an $S^{-1}R$ module $S^{-1}M$ with underlying set

$$S^{-1}M = M \times S/\sim$$

where $(m, s) \sim (m', s')$ if there is a $t \in S$ so that $t(s'm - sm') = 0$. The operations are defined (and proven to be well-defined and satisfy all the module axioms) exactly as in the case of $S^{-1}R$ itself.

This satisfies the following universal property. Let N be an R-module with a map $M \longrightarrow N$. Suppose that for all $s \in S$ we have that $N \longrightarrow N$ via $n \mapsto sn$ is an isomorphism of R-modules. We refer to this as saying that "S acts on N by isomorphisms". Then there is a unique map $S^{-1}M \longrightarrow N$ so that the following diagram commutes

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \\ S^{-1}M \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

where $M \longrightarrow S^{-1}M$ is defined by $m \mapsto \frac{m}{1}$. The proof is exactly like the case of localizing commutative rings.

We connect this to extension of scalars as follows.

Theorem 2.1. *Let $R \longrightarrow S^{-1}R$ be the usual map $r \mapsto \frac{r}{1}$, making $S^{-1}R$ into an R-algebra. Let M be an R-module. Then there is an isomorphism*

$$\alpha_M : S^{-1}M \xrightarrow{\sim} S^{-1}R \otimes_R M$$

In fact, this is "natural" in M in the sense that for any R-linear map $\phi : M \longrightarrow N$ the following diagram commutes

$$\begin{tikzcd} S^{-1}M \arrow[r, "S^{-1}\phi"] \arrow[d, "\alpha_M"'] & S^{-1}N \arrow[d, "\alpha_N"] \\ S^{-1}R \otimes_R S \arrow[r, "\mathrm{id} \otimes \phi"'] & S^{-1}R \otimes_R N \end{tikzcd}$$

Remark. That this isomorphism is natural means that we can essentially freely replace any instance of $S^{-1}M$ with $S^{-1}R \otimes_R M$. If it were not natural, we could run into issues casually making this replacement. For instance, you can run into issues in linear algebra by casually replacing a finite dimensional vector space with its dual. Using Emily Riehl's analogy of objects in category theory as nouns and morphisms as verbs, this isomorphism provides a means of translation between $S^{-1}(-)$ and $S^{-1}R \otimes_R (-)$ which respects the "grammar" rather than just being a "discrete translation" which doesn't understand the relationship between the nouns.

Proof. The philosophy is that these two objects have the same universal property, as an R-module by which S acts via isomorphisms is the same as an $S^{-1}R$-module. This is essentially a proof, but it requires some work to make rigorous. We will at least define the necessary maps.

For one, the $S^{-1}R$-linear map $S^{-1}M \longrightarrow S^{-1}R \otimes_R M$ exists by universal property of $S^{-1}M$, as S acts by isomorphisms on $S^{-1}R \otimes_R M$ due to it being an $S^{-1}R$-module and S mapping to units in $S^{-1}R$. Explicitly, it takes $\frac{m}{s} \mapsto \frac{1}{s} \otimes m$.

On the other hand, $S^{-1}M$ is an $S^{-1}R$-module with a map $M \longrightarrow S^{-1}M$, so the universal property of extension of scalars yields an $S^{-1}R$-linear map $S^{-1}R \otimes_R M \longrightarrow S^{-1}M$ via $\frac{r}{s} \otimes m \mapsto \frac{r}{s}m$.

One can check that these are inverse to one another. Naturality can be checked as well, and intuitively it follows as α_M was constructed with no choices whatsoever. $\square$

I now state some useful facts about localization.

Theorem 2.2 (Exactness of localization)**.** *Let* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be a shrot exact sequence of* R*-modules, and let* $S \subseteq R$ *be a multiplicative subset. Then* $0 \longrightarrow S^{-1}A \longrightarrow S^{-1}B \longrightarrow S^{-1}C \longrightarrow 0$ *is exact.*

Proof. Right exactness follows from right exactness of the tensor product and the natural isomorphism above. The naturality is critical here!

All that remains then is to show that $S^{-1}A \longrightarrow S^{-1}B$ is injective, which can be shown by direct computation. Let $\frac{a}{s} \mapsto 0$... $\square$

This result essentially says that localization and quotients commute with each other. There is a partial converse to this exactness.

Theorem 2.3 (Exactness is a local property)**.** *The following are equivalent for modules* A, B, C *over* R*.*

(i) $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is exact.*

(ii) $0 \longrightarrow A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0$ *is exact for all prime ideals* $\mathfrak{p} \subseteq R$*, where* $A_{\mathfrak{p}}$ *is the localization of* A *by* $R - \mathfrak{p}$*.*

(iii) $0 \longrightarrow A_{\mathfrak{m}} \longrightarrow B_{\mathfrak{m}} \longrightarrow C_{\mathfrak{m}} \longrightarrow 0$ *is exact for all maximal ideals* $\mathfrak{m} \subseteq R$*.*

Proof. Exactness of a sequence $\ldots \longrightarrow A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \longrightarrow \ldots$ means $\ker(f_{i+1})/\operatorname{im}(f_i) = 0$ for all i. Use that localization commutes with quotients, via the exactness above, and the fact that an R-module M is 0 iff $M_{\mathfrak{m}}$ is 0 for all maximal ideals $\mathfrak{m}$. Apply this to the R-module $M = \ker(f_{i+1})/\operatorname{im}(f_i)$. $\square$