1 Extension of Scalars

Let R be a commutative ring. Recall briefly the notion of an R-algebra

Definition 1.1. An *R*-algebra is a map $f : R \longrightarrow S$ where *S* is a commutative ring. This means that *S* is a commutative ring with the structure of an *R*-module via the formula

$$rs := f(r)s$$

for $r \in R$ and $s \in S$.

Since S is an R-module it may be tensored (over R) by other R-modules, which will yield for us extension of scalars. Before we get into that, we first define the pullback or restriction of scalars.

Definition 1.2. Let S be an R-algebra via $f : R \longrightarrow S$. Let M be an S-module. We define an R-module f^*M which is the same underlying abelian group as M equipped with the R-action

$$rm := f(r)m$$

using the existing S-module structure on M. Often, we will be lazy and suppress the f^* from the notation, and simultaneously refer to M as an S-module and as an R-module via this formula.

For example, let V be a vector space over \mathbb{C} . Consider the inclusion map $i : \mathbb{R} \longrightarrow \mathbb{C}$. This makes \mathbb{C} into an \mathbb{R} -algebra, so we can restrict scalars from \mathbb{C} to \mathbb{R} to get a vector space i^*V over \mathbb{R} . This is the complex vector space V viewed as a real vector space in the usual way. Hence, we have restricted the scalars from \mathbb{C} to \mathbb{R} . This works the same way for any field extension.

Now we go the other way, which requires the tensor product.

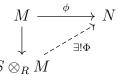
Definition 1.3. Let $f : R \longrightarrow S$ be an *R*-algebra and *M* be an *R*-module. We consider the *R*-module $S \otimes_R M$, sometimes called f_*M , and equip it with an *S*-module structure via

$$s(s' \otimes m) = (ss') \otimes m$$

To be a bit more formal, fix $s \in S$ and define the *R*-bilinear map $S \times M \longrightarrow S \otimes_R M$ via $(s', m) \mapsto (ss') \otimes m$. Then use the universal property of the tensor product to define the map as above.

Now there is an induced R-linear map $M \longrightarrow S \otimes_R M$ via $m \mapsto 1 \otimes m$. We claim that this is the "most efficient" way to transform M from an R-module into an S-module. Formally, we assert the following universal property.

Theorem 1.1. Let S be an R-algebra via $f : R \longrightarrow S$ and let N be an S-module and suppose there is an R-linear map $\phi : M \longrightarrow N$. Then there is a unique S-linear map $\Phi : S \otimes_R M \longrightarrow N$ so that the following diagram commutes



Remarks. (i) We said N was an S-module and then asserted that $\phi : M \longrightarrow N$ was R-linear. This is meant to be read as saying that N is an R-module from the restriction of scalars via $f: R \longrightarrow S$. So we could have said let $\phi : M \longrightarrow f^*N$ be R-linear, but this is cumbersome so we choose to suppress it. (ii) This is remarkably close to the universal property of localization! This is not an accident philosophically or literally. On a philosophical level, this sort of universal property is what we expect from the "most efficient" way to perform a construction. Both universal properties are examples of an "adjunction" in category theory. On a literal level, we will show later that localization of modules is an example of extension of scalars.

Proof. We construct the map Φ . Indeed, we define it via the universal property of the tensor product by sending

$$s \otimes m \mapsto s\phi(m)$$

where, as usual, we mean to take the *R*-bilinear map $S \times M \longrightarrow N$ via

$$(s,m) \mapsto s\phi(m)$$

and induce the map $\Phi: S \otimes_R M \longrightarrow N$ by universal property.

Let's check that this is *R*-bilinear. By ϕ being a group homomorphism and distributivity of multiplication, it is surely biadditive (aka \mathbb{Z} -bilinear). So we check that this map respects *R* scalars. First of all,

$$(rs,m) \mapsto (rs)\phi(m)$$

= $(f(r)s)\phi(m)$
= $f(r)(s\phi(m))$
= $r(s\phi(m))$

The second line is by definition of the R-module structure on S via f, and the last line is by definition of the R-module structure on N via restriction of scalars along f. On the other hand, we have

$$(s, rm) \mapsto s\phi(rm)$$

= $s(r\phi(m))$
= $s(f(r)\phi(m))$
= $(sf(r))\phi(m)$
= $(rs)\phi(m)$
= $r(s\phi(m))$

Hence, this map is *R*-bilinear so Φ is *R*-linear by universal property of the tensor product.

Furthermore, we see that $\Phi(1 \otimes m) = 1\phi(m) = \phi(m)$, so the diagram does indeed commute.

Now, we have constructed an R-linear map $\Phi : S \otimes_R M \longrightarrow N$ which commutes in the diagram, so we now want to prove that it is in fact S-linear. Indeed, we check this on simple tensors, which are generators, so we compute

$$\Phi(s(s' \otimes m)) = \Phi((ss') \otimes m)$$
$$= (ss')\phi(m)$$
$$= s(s'\phi(m))$$
$$= s\Phi(s' \otimes m)$$

so we have shown S-linearity of Φ .

Finally, we show that Φ is unique. Suppose there was an S-linear map $\Psi: S \otimes_R M \longrightarrow N$ which commuted in this diagram, i.e.

$$\Psi(1\otimes m) = \phi(m)$$

Then for any $s \in S$, we have by S-linearity that

$$\Psi(s\otimes m) = \Psi(s(1\otimes m)) = s\Psi(1\otimes m) = s\phi(m) = \Phi(s\otimes m)$$

So as simple tensors generate $S \otimes_R M$, we have shown $\Psi = \Phi$, proving uniqueness.

Remark. Extension of scalars is "functorial". That is, given an *R*-linear map $f : M \longrightarrow N$ we have the induced map $\operatorname{id} \otimes f : S \otimes_R M \longrightarrow S \otimes_R N$ which is *S*-linear.

Examples. (i) Let V be a vector space over \mathbb{R} . As before, the inclusion $i : \mathbb{R} \longrightarrow \mathbb{C}$ makes \mathbb{C} into an \mathbb{R} -algebra. Then $\mathbb{C} \otimes_{\mathbb{R}} V$ is the extension of scalars of V to \mathbb{C} . If V has basis $\{\beta_i\}_{i \in I}$ then we may write

$$V \cong \coprod_{i \in I} \mathbb{R}\beta_i$$

We have that the tensor product commutes with the coproduct, so

$$\mathbb{C} \otimes_{\mathbb{R}} V \cong \prod_{i \in I} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}\beta_i$$

and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}\beta_i$ is a one dimensional \mathbb{C} vector space with basis $1 \otimes \beta_i$. Thus, we conclude that $\mathbb{C} \otimes_{\mathbb{R}} V$ is the complex vector space with basis $\{1 \otimes \beta_i\}_{i \in I}$. Essentially, we have just replaced \mathbb{R} with \mathbb{C} .

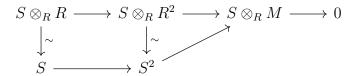
(ii) Suppose we have an *R*-module with an explicit presentation like $M = R\langle x_1, x_2 | 2x_1 = x_2 \rangle$. Formally, this means that there is an exact sequence

$$\begin{array}{cccc} R & \longrightarrow & R^2 & \longrightarrow & M & \longrightarrow & 0 \\ 1 & \longmapsto & (2, -1) & & \\ & & e_i & \longmapsto & x_i \end{array}$$

Now, let $R \longrightarrow S$ be an *R*-algebra. We tensor this sequence by *S* to get a presentation of $S \otimes_R M$. Indeed, by right exactness of the tensor product, the following is an exact sequence of *S*-modules.

$$S \otimes_R R \longrightarrow S \otimes_R R^2 \longrightarrow S \otimes_R M \longrightarrow 0$$

We have isomorphisms $S \otimes_R R \longrightarrow S$ and $S \otimes_R R^2 \longrightarrow S^2$ via $s \otimes r \mapsto sr$ and $s \otimes (a, b) \mapsto (sa, sb)$. We can thus form



We compute the image of 1 in $S \longrightarrow S^2$ via commutativity of this square. Indeed,

$$1 \otimes 1 \longmapsto 1 \otimes (2, -1)$$

$$\uparrow \qquad \qquad \downarrow$$

$$1 \qquad \qquad (2, -1)$$

so this is given by $1 \mapsto (2, -1)$. Similarly, $S^2 \longrightarrow S \otimes_R M$ is given by $e_i \mapsto 1 \otimes x_i$. Write $y_i = 1 \otimes x_i$.

We conclude that we have the presentation

$$S \otimes_R M = S\langle y_1, y_2 | 2y_1 = y_2 \rangle$$

Thus, when given a presentation of a module, we extend scalars by just replacing R with S.

2 Localization

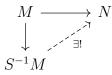
As discussed previously, there is a close relationship between localization and extension of scalars, which we now explore. For one, both localization and extension of scalars are ways to turn R-modules into $S^{-1}R$ -modules. Let's first recall localization of modules.

Let R be a commutative ring with a multiplicative subset S. For an R-module M we define an $S^{-1}R$ module $S^{-1}M$ with underlying set

$$S^{-1}M = M \times S/\sim$$

where $(m, s) \sim (m', s')$ if there is a $t \in S$ so that t(s'm - sm') = 0. The operations are defined (and proven to be well-defined and satisfy all the module axioms) exactly as in the case of $S^{-1}R$ itself.

This satisfies the following universal property. Let N be an R-module with a map $M \longrightarrow N$. Suppose that for all $s \in S$ we have that $N \longrightarrow N$ via $n \mapsto sn$ is an isomorphism of R-modules. We refer to this as saying that "S acts on N by isomorphisms". Then there is a unique map $S^{-1}M \longrightarrow N$ so that the following diagram commutes



where $M \longrightarrow S^{-1}M$ is defined by $m \mapsto \frac{m}{1}$. The proof is exactly like the case of localizing commutative rings.

We connect this to extension of scalars as follows.

Theorem 2.1. Let $R \longrightarrow S^{-1}R$ be the usual map $r \mapsto \frac{r}{1}$, making $S^{-1}R$ into an R-algebra. Let M be an R-module. Then there is an isomorphism

$$\alpha_M: S^{-1}M \xrightarrow{\sim} S^{-1}R \otimes_R M$$

In fact, this is "natural" in M in the sense that for any R-linear map $\phi: M \longrightarrow N$ the following diagram commutes

$$\begin{array}{ccc} S^{-1}M & \xrightarrow{S^{-1}\phi} & S^{-1}N \\ & & & & \downarrow^{\alpha_N} \\ & & & \downarrow^{\alpha_N} \\ S^{-1}R \otimes_R S & \xrightarrow{\mathrm{id} \otimes \phi} & S^{-1}R \otimes_R N \end{array}$$

Remark. That this isomorphism is natural means that we can essentially freely replace any instance of $S^{-1}M$ with $S^{-1}R \otimes_R M$. If it were not natural, we could run into issues casually making this replacement. For instance, you can run into issues in linear algebra by casually replacing a finite dimensional vector space with its dual. Using Emily Riehl's analogy of objects in category theory as nouns and morphisms as verbs, this isomorphism provides a means of translation between $S^{-1}(-)$ and $S^{-1}R \otimes_R(-)$ which respects the "grammar" rather than just being a "discrete translation" which doesn't understand the relationship between the nouns.

Proof. The philosophy is that these two objects have the same universal property, as an R-module by which S acts via isomorphisms is the same as an $S^{-1}R$ -module. This is essentially a proof, but it requires some work to make rigorous. We will at least define the necessary maps.

For one, the $S^{-1}R$ -linear map $S^{-1}M \longrightarrow S^{-1}R \otimes_R M$ exists by universal property of $S^{-1}M$, as S acts by isomorphisms on $S^{-1}R \otimes_R M$ due to it being an $S^{-1}R$ -module and S mapping to units in $S^{-1}R$. Explicitly, it takes $\frac{m}{s} \mapsto \frac{1}{s} \otimes m$.

On the other hand, $S^{-1}M$ is an $S^{-1}R$ -module with a map $M \longrightarrow S^{-1}M$, so the universal property of extension of scalars yields an $S^{-1}R$ -linear map $S^{-1}R \otimes_R M \longrightarrow S^{-1}M$ via $\frac{r}{s} \otimes m \mapsto \frac{r}{s}m$.

One can check that these are inverse to one another. Naturality can be checked as well, and intuitively it follows as α_M was constructed with no choices whatsoever.

I now state some useful facts about localization.

Theorem 2.2 (Exactness of localization). Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a shrot exact sequence of *R*-modules, and let $S \subseteq R$ be a multiplicative subset. Then $0 \longrightarrow S^{-1}A \longrightarrow S^{-1}B \longrightarrow S^{-1}C \longrightarrow 0$ is exact.

Proof. Right exactness follows from right exactness of the tensor product and the natural isomorphism above. The naturality is critical here!

All that remains then is to show that $S^{-1}A \longrightarrow S^{-1}B$ is injective, which can be shown by direct computation. Let $\frac{a}{s} \mapsto 0...$

This result essentially says that localization and quotients commute with each other. There is a partial converse to this exactness.

Theorem 2.3 (Exactness is a local property). The following are equivalent for modules A, B, C over R.

- (i) $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact.
- (ii) $0 \longrightarrow A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0$ is exact for all prime ideals $\mathfrak{p} \subseteq R$, where $A_{\mathfrak{p}}$ is the localization of A by $R \mathfrak{p}$.
- (iii) $0 \longrightarrow A_{\mathfrak{m}} \longrightarrow B_{\mathfrak{m}} \longrightarrow C_{\mathfrak{m}} \longrightarrow 0$ is exact for all maximal ideals $\mathfrak{m} \subseteq R$.

Proof. Exactness of a sequence $\ldots \longrightarrow A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \longrightarrow \ldots$ means $\ker(f_{i+1})/\operatorname{im}(f_i) = 0$ for all i. Use that localization commutes with quotients, via the exactness above, and the fact that an R-module M is 0 iff $M_{\mathfrak{m}}$ is 0 for all maximal ideals \mathfrak{m} . Apply this to the R-module $M = \ker(f_{i+1})/\operatorname{im}(f_i)$.