Tensor products part 2

Recall "right exactness" of the tensor product.

Thm. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of R-Modules. Let M be an R-Module. Then

$$A \otimes_R M \xrightarrow{f \otimes id} B \otimes_R M \xrightarrow{g \otimes id} C \otimes_R M \to 0$$

is exact

P.S. <u>Exactness at $C \otimes_R M$</u>

We want to show $g \otimes id$ is onto.

Let $c \otimes m \in C \otimes_R M$. As g is onto (by exactness at C) $\exists b \in B$ s.t. $g(b) = c$. Then $(g \otimes id)(b \otimes m) = g(b) \otimes id(m)$
$= c \otimes m$

so $c \otimes m \in im(g \otimes id)$. These generate $C \otimes_R M$, so $g \otimes id$ is onto

Exactness at $B \otimes_R M$

$$(g \otimes id)\big((f \otimes id)(a \otimes m)\big) = (g \otimes id)(f(a) \otimes m)$$
$$= g(f(a)) \otimes m$$
$$= 0 \otimes m$$
$$= 0$$

(Rmk. $g \otimes id \circ f \otimes id = g \circ f \otimes id$)

Thus, $\mathrm{im}(f \otimes id) \subseteq \ker(g \otimes id)$

OTOH we want to show $\ker(g \otimes id) = \mathrm{im}(f \otimes id)$.

Directly letting $g \otimes id\left(\sum r_i(b_i \otimes m_i)\right) = 0$ is hard.

Instead, consider

$$\frac{B \otimes_R M}{\mathrm{im}(f \otimes id)} \longrightarrow C \otimes_R M$$

The kernel is $\dfrac{\ker(g \otimes id)}{\mathrm{im}(f \otimes id)}$, so it suffices to show this map is injective

We define an inverse

$$C \otimes_R M \longrightarrow \frac{B \otimes_R M}{\operatorname{im}(f \otimes id)}$$

by UP as such.

For $c \otimes m \in C \otimes_R M$, let $b \in B$ s.t. $g(b) = c$.

Then let

$$c \otimes m \longmapsto b \otimes m + \operatorname{im}(f \otimes id)$$

Note, if $g(b') = c$ as well, then $b - b' \in \ker(g) = \operatorname{im}(f)$,

so $b \otimes m - b' \otimes m \in \operatorname{im}(f \otimes id)$

Thus, this is independent of the choice of b.

We compute the composition

$$\frac{B \otimes_R M}{\operatorname{im}(f \otimes id)} \xrightarrow{\alpha} C \otimes_R M \xrightarrow{\beta} \frac{B \otimes_R M}{\operatorname{im}(f \otimes id)}$$

$$b \otimes m + \operatorname{im}(f \otimes id) \longmapsto g(b) \otimes m \longmapsto b \otimes m + \operatorname{im}(f \otimes id)$$

so $\beta \circ \alpha = id$, thus α is injective. $\square$

We also had left exactness of Hom

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$$

is exact w/ the same setup.

Pf. exc (it's easier than $\otimes$).

Right exactness says, in a sense, that $\otimes$ commutes w/ quotients.

Example.

$$\left(0 \to I \to R \to R/I \to 0\right) \otimes_R M$$

$$\parallel$$

$$\begin{tikzcd}
I \otimes_R M \arrow[r] \arrow[dr, "\alpha"'] & R \otimes_R M \arrow[r] \arrow[d, "\sim"] & R/I \otimes_R M \arrow[r] \arrow[d, "\sim"] & 0 \\
& M \arrow[r] & M/\mathrm{im}(\alpha) &
\end{tikzcd}$$

$r \otimes m \mapsto rm$

$(r+I) \otimes m \mapsto rm + IM$

$x \otimes m \mapsto x \cdot m$

$$M/\mathrm{im}(\alpha) = M/IM$$

e.g.

$$\left(0 \longrightarrow n\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0\right) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$$

$$\rightsquigarrow \quad \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq \frac{\mathbb{Z}/m\mathbb{Z}}{n(\mathbb{Z}/m\mathbb{Z})}$$

Let $g = \gcd(n,m)$.

Then $n(\mathbb{Z}/m\mathbb{Z}) = g\mathbb{Z}/m\mathbb{Z}$

$$\therefore \frac{\mathbb{Z}/m\mathbb{Z}}{n(\mathbb{Z}/m\mathbb{Z})} = \frac{\mathbb{Z}/m\mathbb{Z}}{g\mathbb{Z}/m\mathbb{Z}} \simeq \mathbb{Z}/g\mathbb{Z}$$

i.e. $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/(m,n)\mathbb{Z}$.

We know also that $\otimes$ commutes w/ $\coprod$

Prop. $\left(\coprod_{i\in I} M_i\right) \otimes N \longrightarrow \coprod_{i\in I} M_i \otimes N$ is an isomorphism

$(m_i)_{i\in I} \otimes n \longmapsto (m_i \otimes n)_{i\in I}$

with inverse given by UP $\coprod$ on $M_i \otimes N \longrightarrow \left(\coprod_{i\in I} M_i\right) \otimes N$ the natural inclusions

Thus, $\mathbb{Z}$, $\mathbb{Q}$, know how to $\otimes$ any f.g. $\mathbb{Z}$-Mod

(works for any PID really).

A similar computation shows

$$\frac{R[x_1, \ldots, x_n]}{(I)} \otimes_{R[x_1, \ldots, x_n]} \frac{R[x_1, \ldots, x_n]}{J}$$

$$\cong ?$$

$$\frac{R[x_1, \ldots, x_n]}{(I + J)}$$

So this is (up to a radical) the coordinate ring

of $V(I) \cap V(J)$ in $\mathbb{A}^n_R$.

Prop. Let V, W be k-modules for k a field. Let V be finite dimensional. Then $V^* \otimes_k W \cong \mathrm{Hom}_k(V, W)$ (naturally in V and W)

Pf. Let
$$\begin{aligned} V^* \otimes W &\longrightarrow \mathrm{Hom}_k(V,W) \\ f \otimes w &\longmapsto (v \mapsto f(v)w) \end{aligned}$$

We claim this is an isomorphism.

Let $v_1, \dots, v_n$ be a basis for V and let $v_1^*, \dots, v_n^*$ be the dual basis for V^*

Define
$$\begin{aligned} \mathrm{Hom}_k(V,W) &\longrightarrow V^* \otimes W \\ T &\longmapsto \sum v_i^* \otimes T(v_i) \end{aligned}$$

$$\begin{aligned} V^* \otimes W &\longrightarrow \mathrm{Hom}_k(V,W) \longrightarrow V^* \otimes W \\ v_i^* \otimes w &\longmapsto (v \mapsto v_i^*(v)w) \longmapsto \sum v_j^* \otimes (v_i^*(v_j)w) \\ &\qquad\qquad\qquad\qquad = v_i^* \otimes w \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_k(V,W) &\longrightarrow V^* \otimes W \longrightarrow \mathrm{Hom}_k(V,W) \\ T &\longmapsto \sum v_i^* \otimes T(v_i) \longmapsto \left(v \mapsto \sum v_i^*(v)\, T(v_i) \atop \qquad = T\left(\sum v_i^*(v) v_i\right) = T(v) \right) \end{aligned}$$

$\square$

Rmk. - This recapitulates the decomposition of matrices into sums of rank one matrices

$A \in M_{n\times m}(k)$, then $A = \sum A(e_i) e_i^t$

- If V is ∞ dim, this map is 1-1 w/ image the finite rank maps $V \to W$

Application. Let $\dim_k V < \infty$. Consider $V^* \otimes V \cong \mathrm{End}_k(V)$.

Then $V^* \otimes V \xrightarrow{ev} k$ naturally exists.

$f \otimes v \longmapsto f(v)$

$$\mathrm{End}_k(V) \xrightarrow{\sim} V^* \otimes V \xrightarrow{ev} k$$
$$T \longmapsto \sum v_i^* \otimes T(v_i) \longmapsto \sum v_i^*(T(v_i))$$

If $T(v_i) = \sum_j a_{ij} v_j$ then

$$\sum_i v_i^*(T(v_i)) = \sum_i v_i^*\left(\sum_j a_{ij} v_j\right)$$
$$= \sum_i a_{ii}$$
$$= tr(T)$$

So $ev \longleftrightarrow$ trace under this iso!

Rmk. — $\mathrm{End}_{\mathbb{R}}(V)$ has a special element id

$$\begin{array}{ccc} \mathrm{End}_{\mathbb{R}}(V) & \longrightarrow & V^* \otimes V \\ id & \longmapsto & \underbrace{\sum v_i^* \otimes v_i} \end{array}$$

This is called
the Casimir element

Thus, the Casimir element is basis independent

- A physicist calls an element of

$$V^{\otimes p} \otimes (V^*)^{\otimes q}$$

a (p,q) tensor

$(1,0)$ = vector
$(0,1)$ = functional (covector)
$(1,1)$ = , as above, a linear operator
The cross product is a $(1,2)$ tensor)

(really, a (p,q)-tensor is a map $(V^*)^{\otimes p} \otimes V^{\otimes q} \longrightarrow \mathbb{R}$)

Extension of Scalars and $\otimes$ of algebras

Let $R \xrightarrow{f} S$. Then S is naturally an R-Module via

$$r \cdot s = f(r)s$$

We say S is an R-algebra

Let M be an R-Module

Prop. $S \otimes_R M$ is an S-Module via

$$s(s' \otimes m) = (ss') \otimes m$$

f induces a map

$$\begin{tikzcd} R \otimes_R M \arrow[r, "f \otimes id"] & S \otimes_R M \\ M \arrow[u, "\sim", "1 \otimes m"'] \arrow[ur, bend right, "1 \otimes m"'] & \end{tikzcd}$$

$$m \mapsto 1 \otimes m$$

Satisfying the UP that is $M \xrightarrow{\varphi} N$ is R-linear with N an S-module, made into an R-module via pullback along f
$r \cdot n = f(r)n$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \\ S \otimes_R M \arrow[ur, dashed, "\exists!\ S\text{-linear}"'] & \end{tikzcd}$$

pf. We show the UP.

First, consider

$$S \times M \longrightarrow N$$
$$(s, m) \longmapsto s\varphi(m)$$

this is R-bilinear. It's surely biadditive ($\mathbb{Z}$-bilinear)

$$(rs, m) \longmapsto (rs)\varphi(m)$$
$$= f(r)\, s\varphi(m)$$
$$= r(s(\varphi(m)))$$

$$(s, rm) \longmapsto s\varphi(rm)$$
$$= s(r\varphi(m))$$
$$= s(f(r)\varphi(m))$$
$$= r(s(\varphi(m)))$$

So this yields a map $S \otimes_R M \xrightarrow{\Phi} N$ commuting in the diagram. As for S-linearity,

$$\Phi(s(s' \otimes m)) = \Phi((ss') \otimes m) = ss'\varphi(m)$$
$$= s(s'\varphi(m))$$
$$= s\Phi(s' \otimes m)$$

$\square$

$S \otimes_R M$ is called the extension of scalars of M to S (via f),

e.g., Let K/F a field extension, V a vector space over F w/ basis $\{\beta_i\}_{i\in I}$,

Then $V \otimes_F K$ is a K-vs w/ basis $\{\beta_i \otimes 1\}_{i \in I}$

Pf. $V \otimes_F K \cong \left(\coprod_{i\in I} F \cdot \beta_i\right) \otimes_F K \cong \coprod_{i \in I} (F\cdot\beta_i) \otimes_F K$

$$\cong \coprod_{i\in I} K(\beta_i \otimes 1)$$

e.g. Let M be an R-module given by generators and relations, like

$$M = R\langle e_1, e_2 \mid 2e_1 = e_2\rangle$$

Then $M \otimes_R S \cong S\langle e_1, e_2 \mid 2e_1 = e_2\rangle$

indeed, this presentation means (by definition)

$$\begin{array}{ccccc} R & \longrightarrow & R^2 & \longrightarrow & M \longrightarrow 0 \\ 1 & \longmapsto & \begin{pmatrix} 2 \\ -1 \end{pmatrix} & & \end{array}$$

Tensor by S and use right exactness to get

$$\begin{tikzcd}
R \otimes_R S \arrow[r] \arrow[d, "\sim"'] & R^2 \otimes_R S \arrow[r] \arrow[d, "\sim"'] & M \otimes_R S \arrow[r] & 0 \\
S \arrow[r] & S^3 \arrow[ur] & & \\
1 \arrow[r, mapsto] & \begin{pmatrix} 3 \\ -1 \end{pmatrix} & &
\end{tikzcd}$$

Prop. Let
$$\begin{tikzcd}
R \arrow[r, "f_1"] \arrow[d, "f_2"'] & S_1 \\
S_2 &
\end{tikzcd}$$
be two R-algebras. Then

$S_1 \otimes_R S_2$ is an R-algebra via the multiplication

$$(s_1 \otimes s_2)(s_1' \otimes s_2') := (s_1 s_1') \otimes (s_2 s_2')$$

There are then maps

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S_1 \arrow[d] & s_1 \arrow[d, mapsto] \\
S_2 \arrow[r] & S_1 \otimes_R S_2 & s_1 \otimes 1 \\
s_2 \arrow[r, mapsto] & 1 \otimes s_2 &
\end{tikzcd}$$

satisfying the following UP

If $R \xrightarrow{g} A$ is an R-algebra with maps

$S_i \xrightarrow{g_i} A \quad i=1,2$ so that

$$\begin{tikzcd}
R \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, "g"] & S_1 \arrow[d, "g_1"] \\
S_2 \arrow[r, "g_2"'] & A
\end{tikzcd}$$

commutes, i.e. $g_2 f_2 = g_1 f_1 = g$

Then

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S_1 \arrow[d] \arrow[ddr, bend left] & \\
S_2 \arrow[r] \arrow[drr, bend right] & S_1 \otimes_R S_2 \arrow[dr, dashed, "\exists !"] & \\
& & A
\end{tikzcd}$$

w/ $S_1 \otimes_R S_2 \longrightarrow A$ a map of R-algebras

we say $S_1 \otimes_R S_2$ is the coproduct of S_1 and S_2 as R-algebras pf. We construct the map

$$S_1 \otimes_R S_2 \longrightarrow A$$

$$s_1 \otimes s_2 \longmapsto q_1(s_1)\, q_2(s_2)$$

by UP $\otimes$.