¿Is heart the where is tensor

Let R be a commutative ring.

Def. Let M, N, T be R-modules. A function

$$f : M \times N \longrightarrow T$$

is called R-bilinear if

$$\left.\begin{aligned} f(m_1 + m_2, n) &= f(m_1, n) + f(m_2, n) \\ f(m, n_1 + n_2) &= f(m, n_1) + f(m, n_2) \\ f(rm, n) &= r f(m, n) \\ f(m, rn) &= r f(m, n) \end{aligned}\right] \text{"distributivity"}$$

i.e. $f(m, -)$ and $f(-, n)$ are R-linear.

e.g. - $R \times R \longrightarrow R$ $\quad (r_1, r_2) \longmapsto r_1 r_2$ is R-bilinear

- $R \times M \longrightarrow M$ $\quad (r, m) \longmapsto rm$

- V a vs$/\mathbb{R}$ w/ an inner product $\langle , \rangle$. Then $\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$ is $\mathbb{R}$-bilinear

- $V^* \times V \longrightarrow \mathbb{R}$ $\quad (f, v) \longmapsto f(v)$

- $\mathrm{Hom}(M,N) \times \mathrm{Hom}(T,M) \longrightarrow \mathrm{Hom}(T,N)$

  $(f,g) \longmapsto f \circ g$

- $M_{n\times m}(R) \times M_{m\times r}(R) \longrightarrow M_{n\times r}(R)$

  $(A,B) \longmapsto AB$

- $M_n(R) \times M_n(R) \longrightarrow M_n(R)$

  $(A,B) \longmapsto AB-BA \ (=[A,B])$

- $R^2 \times R^2 \xrightarrow{\det} R$

  $\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) \longmapsto ad-bc$

we also have trilinear, quadlinear, . . .

we want to make bilinear maps linear via formal multiplication.

Def. Let M, N be R-modules. A tensor product of M and N over R is an R-module T with an R-bilinear map $M\times N \longrightarrow T$ s.t. for any bilinear map $M \times N \longrightarrow U$ we have a unique linear map "$M \otimes_R N$" $\longrightarrow T$ s.t.

$$\begin{tikzcd} M\times N \arrow[r] \arrow[d] & M\times N \\ T \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

commutes.

Currently, we don't know that T exists.

Prop. If T_1, T_2 are two tensor products of M and N (over R) then they're isomorphic via a unique isomorphism.

Pf. We have

$$\begin{tikzcd} M \times N \arrow[r, "\text{bilinear}"] \arrow[d, "\text{bilinear}"'] & T_1 \arrow[dl, dashed, bend left, "\exists!\ f_1"] \\ T_2 \arrow[ur, dashed, bend left, "\exists!\ f_2"] & \end{tikzcd}$$

Furthermore,

$$\begin{tikzcd} M \times N \arrow[r] \arrow[d] & T_1 \\ T_1 \arrow[ur, "f_2 f_1"'] & \end{tikzcd}$$

commutes, so by uniqueness $f_2 f_1 = id_{T_1}$. Similarly, $f_1 f_2 = id_{T_2}$. The f_i are nec. unique s.t. the diagram commutes. $\square$

c.f. uniqueness of localization!

Now, we construct a (the by our uniqueness) tensor product.

<u>Construction</u>. Consider the free R-module on the <u>set</u> $M \times N$

$$\coprod_{(m,n) \in M \times N} R \cdot (m,n)$$

Its elements are formal combinations

$$\sum_{i=1}^{k} r_i (m_i, n_i)$$

i.e. $M \times N$ is a basis for this space.

Any set map $M \times N \longrightarrow T$ induces an R-linear map $\coprod_{M \times N} R \cdot (m,n) \longrightarrow T$

to restrict this to bilinear maps, which is doable via a quotient

Let $$M \otimes_R N = \coprod_{M \times N} R(m,n) \Big/ \left(\begin{array}{l|l} (m, n_1 + n_2) - ((m,n_1) + (m,n_2)), & r \in R \\ (n, m_1 + m_2) - ((n,m_1) + (n,m_2)), & n, n_1, n_2 \in N \\ (rn, m) - r(n,m), & m, m_1, m_2 \in M \\ (n, rm) - r(n,m) & \end{array} \right)$$

we have the map

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr] & \coprod_{M \times N} R(m,n) \arrow[d] \\ & M \otimes_R N \end{tikzcd}$$

which we denote $(m,n) \longmapsto m \otimes n$.

Thm. $M \otimes_R N$ is a tensor product of M and N (over R)

Pf. By the def of the submodule we quotient by,

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

$$(rm) \otimes n = r(m \otimes n)$$

$$m \otimes (rn) = r(m \otimes n)$$

i.e. $(m,n) \longmapsto m \otimes n$ is R-bilinear

$$M \times N \xrightarrow[\pi]{} M \otimes_R N$$

Now, let $M\times N \xrightarrow{f} T$ be R-bilinear.

$$\begin{tikzcd}
M\times N \arrow[d] \arrow[dr, "f"] & \\
\coprod R(m,n) \arrow[r, dashed, "\exists!\, F"'] & T
\end{tikzcd}$$

$$\sum r_i (m_i, n_i) \longmapsto \sum r_i f(m_i, n_i)$$

Because f is bilinear, $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$,

i.e., $F\big((m, n_1+n_2) - (m, n_1) - (m, n_2)\big) = 0$

Similarly for the other axioms/generators.

Thus, F factors through the quotient

$$\begin{tikzcd}
M\times N \arrow[r, "f"] \arrow[d] & T \\
\coprod R(m,n) \arrow[ur, dashed, "\exists!\, F"] \arrow[d] & \\
M\otimes_R N \arrow[uur, dashed, "\exists!\, \varphi"'] &
\end{tikzcd}$$

$$m\otimes n \longmapsto f(m,n)$$

$\square$

Rmk. We defined

$$m \otimes n \longmapsto f(m,n)$$

but these only generate. $M \times N \to M \otimes_R N$ is not onto, and in general, elts of $M \otimes_R N$ are of the form $\sum r_i (m_i \otimes n_i)$

e.g. – $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$

$$\frac{a}{b} \otimes (\bar{0}) = 0$$

$$\frac{a}{b} \otimes \bar{1} = \frac{2a}{2b} \otimes \bar{1} = 2\left(\frac{a}{2b}\right) \otimes \bar{1} = \frac{a}{2b} \otimes 2 \cdot \bar{1} = \frac{a}{2b} \otimes \bar{0} = 0$$

– $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$

$$\left(\frac{a}{b} + \mathbb{Z}\right) \otimes \left(\frac{c}{d} + \mathbb{Z}\right) = d\left(\frac{a}{bd} + \mathbb{Z}\right) \otimes \left(\frac{c}{d} + \mathbb{Z}\right)$$

$$= \left(\frac{a}{bd} + \mathbb{Z}\right) \otimes \underbrace{(c + \mathbb{Z})}_{0}$$

$$0$$

We rephrase the UP of the tensor product.

Thm (Hom - $\otimes$ adjunction).

$$\operatorname{Hom}_R(M \otimes_R N, T) \xrightarrow{\sim} \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, T)) \quad \text{iso}$$
$$f \longmapsto (m \mapsto (n \mapsto f(m \otimes n)))$$

Pf. By UP $\otimes$, $R\text{-Bil}(M \times N, T) \xleftrightarrow{\sim} \operatorname{Hom}_R(M \otimes_R N, T)$ iso

$$f \longmapsto (m \otimes n \mapsto f(m, n))$$
$$\varphi \circ \pi \longleftarrow\!\shortmid\ \varphi$$

Furthermore,

$$R\text{-Bil}(M \times N, T) \xleftrightarrow{\sim} \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, T)) \quad \text{iso}$$
$$f \longmapsto (m \mapsto (n \mapsto f(m, n)))$$
$$(m, n) \mapsto g(m)(n) \longleftarrow\!\shortmid\ g$$

$\square$

Rmk. "adjunction" is as $\langle Tv, w\rangle = \langle v, T^* w\rangle$.

Let $T(M) = M \otimes_R N$, $H(T) = \operatorname{Hom}_R(N, T)$

$$\operatorname{Hom}_R(T(M), T) = \operatorname{Hom}_R(M, H(T))$$

Properties

i) $M \otimes_R N \xleftrightarrow{\sim} N \otimes_R M$

$m \otimes n \longmapsto n \otimes m$

$m \otimes n \longleftarrow\!\shortmid n \otimes m$

ii) $(M \otimes_R N) \otimes_R T \xleftrightarrow{\sim} M \otimes_R (N \otimes_R T)$

$(m \otimes n) \otimes t \longmapsto m \otimes (n \otimes t)$

we write this as $M \otimes_R N \otimes_R T$, which has the

UP of trilinear maps $M \times N \times T \longrightarrow U$

iii) $\left(\coprod_{i \in I} M_i\right) \otimes N \xleftrightarrow{\sim} \coprod_{i \in I} (M_i \otimes N)$

Compare this to $\mathrm{Hom}_R(N, \prod_{i \in I} M_i) \cong \prod_{i \in I} \mathrm{Hom}_R(N, M_i)$

iv) (functoriality) Let $M_i \xrightarrow{f_i} N_i$, $i = 1, 2$. We get a map

$f_1 \otimes f_2 : M_1 \otimes M_2 \longrightarrow N_1 \otimes N_2$ via

$m_1 \otimes m_2 \longmapsto f_1(m_1) \otimes f_2(m_2)$

Thm (right exactness of $\otimes$). Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$

be a SES of R-modules, and M an R-module.

Then $A \otimes M \xrightarrow{f \otimes id} B \otimes M \xrightarrow{g \otimes id} C \otimes M \longrightarrow 0$ is exact

Pf. Exactness @ $C \otimes M$.

As g is onto, any $c \in C$ is $g(b)$.

Then $c \otimes m = g(b) \otimes m = (g \otimes id)(b \otimes m)$

Exactness at $B \otimes M$

$(g \otimes id) \circ (f \otimes id) = (g \circ f) \otimes id$

$= 0 \otimes id$

$= 0$

So $im(f \otimes id) \subseteq ker(g \otimes id)$.

OTOH let $(g \otimes id)(\sum r_i (b_i \otimes m_i)) = 0$

$\sum r_i \, g(b_i) \otimes m_i = 0$