Hom is where the heart is

Let R be a ring.

Let M and N be R-modules.

Recall $\mathrm{Hom}_R(M,N) = \{ f : M \to N \mid f \text{ is } R\text{-linear} \}$

(R-$\underline{\mathrm{Mod}}\,(M,N)$ in my category-theoretic notation)

Fact. $\mathrm{Hom}_R(M,N)$ is an abelian group under pointwise addition

$$(f+g)(m) := f(m) + g(m)$$

If R is commutative, it's an R-module under pointwise scalar mult.

$$(rf)(m) := r f(m)$$

Rmk. If R is not commutative, rf is not R-linear!

$$\begin{array}{ccc} (rf)(sm) & = r(f(sm)) & = rs\,f(m) \\ (?) \parallel & & \text{?} \parallel \\ s\cdot(rf)(m) & = sr\,f(m) & \end{array}$$

From now onwards, we assume R is commutative.

Prop. "Bilinearity of composition"

Let $M \underset{f'}{\overset{f}{\rightrightarrows}} N \underset{g'}{\overset{g}{\rightrightarrows}} T$ in $R-\underline{Mod}$ and $r \in R$

Then i) $g \circ (f+f') = g \circ f + g \circ f'$

ii) $(g+g') \circ f = g \circ f + g' \circ f$

iii) $(rg) \circ f = r(g \circ f)$

iv) $g \circ (rf) = r(g \circ f)$

we say
$$\begin{aligned} \mathrm{Hom}(N,T) \times \mathrm{Hom}(M,N) &\longrightarrow \mathrm{Hom}(M,T) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

is R-bilinear.

Pf. i). $(g \circ (f+f'))(m) = g((f+f')(m)) = g(f(m) + f'(m))$
$= g(f(m)) + g(f'(m))$
$= (g \circ f)(m) + (g \circ f')(m)$
$= (g \circ f + g \circ f')(m)$

Rest are similar. $\square$

Previously, we had many "universal properties" asserting (natural) bijections on Homs. Often, these will be upgraded to R-linear isomorphisms.

Prop. "Freeness of R"

Let M be an R-module. Define

$$\begin{aligned} \mathrm{Hom}_R(R, M) &\xrightarrow{\varphi} M \\ f &\longmapsto f(1) \end{aligned}$$

This is an isomorphism of R-modules (natural in M).

Pf. φ is clearly R-linear, so we show it's a bijection.

- Suppose $\varphi(f) = 0$. Then $f(1) = 0$. By R-linearity of f, $f(r) = r f(1) = 0$, so $f = 0$.
- Let $m \in M$. We define $f: R \longrightarrow M$ via $f(r) = rm$. This is surely R-linear, and $f(1) = m$. That is, $\varphi(f) = m$. $\square$

Prop. "Universal property of the coproduct"

Let $(M_i)_{i \in I}$, N in R-$\underline{Mod}$. Recall

$$\coprod_{i \in I} M_i = \left\{ (m_i)_{i \in I} \in \prod_{i \in I} M_i \;\middle|\; \begin{array}{c} \text{all but finitely many} \\ m_i \text{ are } 0 \end{array} \right\}$$

Let $\iota_j : M_j \longrightarrow \coprod_{i \in I} M_i$ via

$$m_j \longmapsto (0, \ldots, 0, m_j, 0, \ldots, 0)$$

Then we define

$$\mathrm{Hom}_R\left(\coprod_{i \in I} M_i, N\right) \xrightarrow{\varphi} \prod_{i \in I} \mathrm{Hom}_R(M_i, N)$$

$$f \longmapsto (f \circ \iota_i)_{i \in I}$$

This is an R-linear iso (natural in M_i, N).

Pf. The map is R-linear by bilinearity of composition.

We define an inverse. Suppose we have $(f_i)_{i \in I} \in \prod_{i \in I} \mathrm{Hom}_R(M_i, N)$.

Define $\psi((f_i)_{i \in I}) : \coprod_{i \in I} M_i \longrightarrow N$ via

$$(m_i)_{i \in I} \longmapsto \sum_{i \in I} f_i(m_i)$$

This is well defined as all but finitely many m_i are 0, so the sum is finite.

- $$((\psi \circ \varphi)(f))(m) \quad \psi((f \circ \iota_i)_{i \in I})(m)$$
$$= \sum_{i \in I} f \circ \iota_i(m_i) \qquad \text{where } m = (m_i)_{i \in I}$$

Claim this is $f(m)$.

Indeed, we critically use the identity

$$m = \sum_{i \in I} \iota_i(m_i)$$

Then $$f(m) = f\left(\sum \iota_i(m_i)\right)$$
$$= \sum f(\iota_i(m))$$
$$= \sum (f \circ \iota_i)(m)$$
$$= ((\psi \circ \varphi)(f))(m)$$

So $\psi \circ \varphi = id$

- $$(\varphi \circ \psi)((f_i)_{i \in I}) = \varphi\left(\sum_{i \in I} f_i\right) = \left(\left(\sum_{i \in I} f_i\right) \circ \iota_j\right)_{j \in I}$$

We claim $\left(\sum_{i \in I} f_i\right) \circ \iota_j = f_j$.

Indeed, $\left(\sum_{i\in I} f_i\right)(\iota_j(m))$

$$= \left(\sum_{i\in I} f_i\right)(0,\dots,0,\underset{\substack{\uparrow \\ j^{\text{th}} \text{ position}}}{m},0,\dots)$$

$$= \sum_{i\neq j} f_i(0) + f_j(m)$$

$$= f_j(m)$$

so $\left(\sum f_i\right)\circ\iota_j = f$

Thus, $\varphi\circ\psi = \mathrm{id}$ $\square$

Cor. $\mathrm{Hom}_R\left(\coprod_{i\in I} R, M\right) \cong \prod_{i\in I} M$ via

$$f \longmapsto \sum_{i\in I} f(e^i)$$

where $e^j \in \coprod_{i\in I} R$ is $e^j_i = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{o/w} \end{cases}$

the "j^{th} standard basis vector"

Prop. "Universal property of the product"

$M, (N_i)_{i \in I}$ in R-$\underline{Mod}$. Define

$$\mathrm{Hom}_R\left(M, \prod_{i \in I} N_i\right) \xrightarrow{\varphi} \prod_{i \in I} \mathrm{Hom}_R(M, N_i)$$

$$f \longmapsto (\pi_i \circ f)_{i \in I}$$

where $\pi_j : \prod_{i \in I} N_i \longrightarrow N_j$ is projection

$$(n_i)_{i \in I} \longmapsto n_j$$

This is an R-linear iso (natural in M, N_i).

Pf. Let $\prod_{i \in I} \mathrm{Hom}_R(M, N_i) \xrightarrow{\psi} \mathrm{Hom}_R(M, \prod_{i \in I} N_i)$ via

$$(f_i)_{i \in I} \longmapsto (m \longmapsto (f_i(m))_{i \in I})$$

- $(\varphi \circ \psi)((f_i)_{i \in I}) = (\pi_j \circ (\psi((f_i)_{i \in I})))_{j \in I}$

$$\pi_j \circ (\psi((f_i)_{i \in I})(m))$$

$$= \pi_j((f_i(m))_{i \in I})$$

$$= f_j(m)$$

so $\pi_j \circ \psi((f_i)_{i \in I}) = f_j$. Thus, $\varphi \circ \psi = id$

$$-\ (\psi\circ\varphi)(f) = \psi\big((\pi_i\circ f)_{i\in I}\big)$$

$$m \longmapsto \big((\pi_i\circ f)(m)\big)_{i\in I}$$

$$\|$$

$$(\pi_i(f(m)))_{i\in I}$$

and indeed, for any $n\in\prod_{i\in I} N_i$, $n=(\pi_i(n))_{i\in I}$

so $\psi\circ\varphi=\mathrm{id}$ $\square$

Cor. $\mathrm{Hom}_R(R^n, R^m)\cong M_{m\times n}(R)$

Pf. $\mathrm{Hom}_R(R^n,R^m) \cong \prod_{i=1}^{n}\mathrm{Hom}_R(R,R^m)$ UP $\amalg$

$\cong \prod_{i=1}^{n}\prod_{j=1}^{m}\mathrm{Hom}_R(R,R)$ UP $\amalg$

$\cong \prod_{i=1}^{n}\prod_{j=1}^{m} R$ freeness of R

$\cong M_{m\times n}(R)$

Let $e_1,\dots,e_n$ be the standard basis vectors in R^n. These isomorphisms send

$$f\longmapsto (f\circ\iota_i)_{i=1}^{n}\longmapsto(\pi_j\circ f\circ\iota_i)_{ij}$$

$$\longmapsto(\pi_j(f(e_i)))_{ij}\quad,\ e_i=\iota_i(1)$$

$$\longmapsto\begin{pmatrix}| & & |\\ f(e_1) & \cdots & f(e_n)\\ | & & |\end{pmatrix}$$

$\square$

Homological stuff

Def. - Let $M \xrightarrow{f} N \xrightarrow{g} T$ in $\underline{R\text{-Mod}}$. We say this sequence is "exact" if $\mathrm{im}(f) = \ker(g)$.

- A longer sequence

$$\cdots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \cdots$$

is called exact if each 3-term sequence as above is exact, i.e. $\underbrace{\mathrm{im}(f_i) = \ker(f_{i+1})}_{\text{"exactness at } M_{i+1}\text{"}}$ $\forall i$

e.g. $0 \to M \xrightarrow{f} N$ is exact iff f is injective

$M \xrightarrow{f} N \to 0$ is exact iff f is surjective

$0 \to M \xrightarrow{f} N \to 0$ is exact iff f is an iso

Def. We call an exact sequence

$$0 \to M \xrightarrow{g} N \xrightarrow{f} T \to 0$$

a short exact sequence. M embeds into N via g, and

$T \cong N/\ker(f) = N/\mathrm{im}(g)$ "$= N/M$" $\to$ ($N \not\supseteq M$ in the literal sense) e.g. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

Prop. "Left exactness of Hom"

Let M, A, B, C in R-$\underline{\text{Mod}}$ and

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

a SES.

Then

$$0 \to \operatorname{Hom}_R(M, A) \xrightarrow{f_*} \operatorname{Hom}_R(M, B) \xrightarrow{g_*} \operatorname{Hom}_R(M, C)$$

is exact, where $f_*(\varphi) = f \circ \varphi$ (which are R-linear by bilinearity)

$g_*(\psi) = g \circ \psi$

Pf. - We show exactness at $\operatorname{Hom}_R(M, A)$ i.e. that $\ker(f_*) = 0$.

Let $f_*(\varphi) = 0$, so $f \circ \varphi = 0$. Then $f(\varphi(m)) = 0 \ \forall m \in M$,

so as f is injective, $\varphi(m) = 0 \ \forall m \in M$, thus, $\varphi = 0$.

- We show exactness at $\operatorname{Hom}_R(M, B)$, i.e. $\operatorname{im}(f_*) = \ker(g_*)$

"$\subseteq$" we claim $\mathrm{im}(f_*) \subseteq \ker(g_*)$. Equivalently, that $g_* \circ f_* = 0$. Observe $g_* \circ f_* = (g \circ f)_*$, and $g \circ f = 0$ by exactness at B. Thus, $g_* \circ f_* = 0$.

"$\supseteq$" Let $\psi \in \ker(g_*)$, so $\psi : M \longrightarrow B$ s.t. $g \circ \psi = 0$.

That is, $g(\psi(m)) = 0 \ \forall m \in M$.

So $\psi(m) \in \ker(g) \ \forall m \in M$

$\ker(g) = \mathrm{im}(f)$ by exactness at B.

Thus, $\psi(m) \in \mathrm{im}(f) \ \forall m \in M$, so $\exists\, \alpha(m) \in A$ s.t.

$\psi(m) = f(\alpha(m))$. As f is injective (exactness at A),

this uniquely determines $\alpha : M \longrightarrow A$.

We claim α is R-linear. Indeed,

$$\begin{aligned} f(\alpha(m+m')) &= \psi(m+m') \\ &= \psi(m) + \psi(m') \\ &= f(\alpha(m)) + f(\alpha(m')) \\ &= f(\alpha(m) + \alpha(m')) \end{aligned}$$

So by injectivity, $\alpha(m+m') = \alpha(m) + \alpha(m')$.

Same idea for $\alpha(rm) = r\alpha(m)$. Thus, $\psi = f_*(\alpha)$ $\square$

What about $\mathrm{Hom}_R(M,B) \xrightarrow{g_*} \mathrm{Hom}_R(M,C) \to 0$?

Counterexample: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

$M = \mathbb{Z}/2\mathbb{Z}$. Then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

and $0 \to \mathbb{Z}/2\mathbb{Z}$ can never be onto.

To say g_* is onto means that given $x : M \to C$ there is a $\varphi : M \to B$ s.t. $g \circ \varphi = x$. That is,

$$\begin{tikzcd}
 & M \arrow[dl, dashed, "\exists \varphi"'] \arrow[d, "x"] & \\
B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

If M satisfies this for *all* $B \to C \to 0$ then we say M is *projective* and that $\mathrm{Hom}_R(M, -)$ is *exact*.

e.g. $M = R$ or $\coprod_{i \in I} R$ are projective (in fact, free).

Dually, Let $N, A, B, C \in R\text{-}\underline{\text{Mod}}$ and

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \text{ a SES.}$$

Then

$$0 \to \mathrm{Hom}_R(C, N) \xrightarrow{g^*} \mathrm{Hom}_R(B, N) \xrightarrow{f^*} \mathrm{Hom}_R(A, N).$$

is exact, when $f^*(\varphi) = \varphi \circ f$

$g^*(\psi) = \psi \circ g$

e.g. Consider $A \subseteq B$ a submodule. Then

$$0 \to A \xhookrightarrow{\iota} B \xrightarrow{\pi} B/A \to 0 \text{ is a SES}$$

Then given any $N \in R\text{-}\underline{\text{Mod}}$,

$$0 \to \mathrm{Hom}_R(B/A, N) \xrightarrow{\pi^*} \mathrm{Hom}_R(B, N) \xrightarrow{\iota^*} \mathrm{Hom}_R(A, N) \text{ is exact}$$

$$\varphi \longmapsto \varphi \circ \pi \qquad \psi \longmapsto \psi \circ \iota = \psi|_A$$

Exactness at $\mathrm{Hom}_R(B, N)$ says that maps $B \to N$ vanishing on A ($\ker \iota^*$) are those maps $B \to N$ arising from $B/A \to N$ ($\mathrm{im}\, \pi^*$). This is the UP of quotients!