1. a). The ideal generated by T is defined as the smallest ideal containing T. So, by definition,

$$T \subseteq I \iff \langle T \rangle \subseteq I$$

for an ideal $I \subseteq R$.

Now, $\mathfrak{p} \in V(T) \iff T \subseteq \mathfrak{p}$ (def'n)

$\iff \langle T \rangle \subseteq \mathfrak{p}$ (above)

$\iff \mathfrak{p} \in V(\langle T \rangle)$ (def'n)

b) $\mathfrak{p} \in V(\mathfrak{B}) \iff \mathfrak{B} \subseteq \mathfrak{p}$ (def'n)

$\implies \mathfrak{a} \subseteq \mathfrak{p}$ as $\mathfrak{a} \subseteq \mathfrak{B}$

$\iff \mathfrak{p} \in V(\mathfrak{a})$ (def'n)

c) $\mathfrak{p} \in V(0) \iff 0 \subseteq \mathfrak{p}$, which is always true

d) $\mathfrak{p} \in V(R) \iff R \subseteq \mathfrak{p}$, which is impossible as prime ideals are proper in R

e) $\langle \bigcup \mathfrak{a}_i \rangle = \sum \mathfrak{a}_i$, as both are the smallest ideal containing all $\mathfrak{a}_i$. Thus, (a) yields the first equality,

$$\mathfrak{p} \in V\left(\sum \mathfrak{a}_i\right) \iff \sum \mathfrak{a}_i \subseteq \mathfrak{p}$$
$$\iff \mathfrak{a}_i \subseteq \mathfrak{p} \quad \forall i \in I$$
$$\iff \mathfrak{p} \in V(\mathfrak{a}_i) \quad \forall i \in I$$
$$\iff \mathfrak{p} \in \bigcap V(\mathfrak{a}_i)$$

f) $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a} \implies V(\mathfrak{a}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$

$\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{b} \implies V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$

Thus, $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$

$\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ so $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$

It suffices to show that $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$,

That is, if $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ then $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.

suppose $\mathfrak{a} \not\subseteq \mathfrak{p}$, Then let $a \in \mathfrak{a} - \mathfrak{p}$.

For $b \in \mathfrak{b}$, $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$, so $ab \in \mathfrak{p}$. As $a \notin \mathfrak{p}$

and as $\mathfrak{p}$ is prime, $b \in \mathfrak{p}$. Thus, $\mathfrak{b} \subseteq \mathfrak{p}$.

So we have shown

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$$

g) $\mathfrak{p} \in \operatorname{Spec} B \iff \mathfrak{p} \subseteq B$ prime, and $\varphi^{-1}[\mathfrak{p}]$ is prime,

hence $\varphi^{-1}[\mathfrak{p}] \in \operatorname{Spec} A$

h) $\mathfrak{p} \in (a_\varphi)^{-1}[V(T)] \iff a_\varphi(\mathfrak{p}) \in V(T)$

$\iff \varphi^{-1}[\mathfrak{p}] \in V(T)$

$\iff T \subseteq \varphi^{-1}[\mathfrak{p}]$

$\iff \varphi[T] \subseteq \mathfrak{p}$

$\iff \mathfrak{p} \in V(\varphi[T])$

2. a) Let $\pi : R \longrightarrow R/\mathfrak{a}$.

claim $\sqrt{\mathfrak{a}} = \pi^{-1}[\text{nil}(R/\mathfrak{a})]$

Indeed, $r \in \pi^{-1}[\text{nil}(R/\mathfrak{a})]$

$\iff \pi(r) \in \text{nil}(R/\mathfrak{a})$

$\iff \exists n \geq 1 \quad r^n + \mathfrak{a} = 0 + \mathfrak{a}$

$\iff \exists n \geq 1 \quad r^n \in \mathfrak{a}$

$\iff r \in \sqrt{\mathfrak{a}}$

or mimic the proof that nil(R) is an ideal.

b) This follows from how we proved (a) given the conclusion of (c).

Alternatively, let $ab \in \sqrt{\mathfrak{a}}$. Then there's some $n \geq 1$ s.t. $(ab)^n \in \mathfrak{a}$. So $a^n b^n \in \mathfrak{a}$, whence $a^n \in \sqrt{\mathfrak{a}}$ or $(b^n)^m \in \sqrt{\mathfrak{a}}$ for some $m \geq 1$. So $a \in \sqrt{\mathfrak{a}}$ or $b \in \sqrt{\mathfrak{a}}$.

c) Every zero divisor in $R/\mathfrak{a}$ being nilpotent means that for all $x + \mathfrak{a}$ s.t. $\exists\, y + \mathfrak{a} \in R/\mathfrak{a}$ non zero with $xy + \mathfrak{a} = 0 + \mathfrak{a}$, we have $x^n + \mathfrak{a} = 0 + \mathfrak{a}$ for some $n \geq 1$. Equivalently, if $xy \in \mathfrak{a}$ with $y \notin \mathfrak{a}$ then $x^n \in \mathfrak{a}$ for some $n \geq 1$. This is the definition of $\mathfrak{a}$ being primary.

d) We claim that the primary ideals of a PID R are precisely the ideals of the form (f^n) for f irreducible in R.

Let $xy \in (f^n)$. Then $f^n | xy$. Suppose $x \notin (f^n)$, i.e. $f^n \nmid x$. By unique factorization, $f | y$ so $f^n | y^n$, whence $y^n \in (f^n)$.

Now, we claim all primary ideals are of this form. Indeed, we show (a) is not primary when a is divisible by at least 2 distinct prime factors.

Say f is a prime factor of a, and let $f^e || a$, i.e. f^e is the power of f in the factorization of a. Then let $g = \frac{a}{f^e}$. As a is divisible by multiple factors, g is not a unit. We have $fg \in (a)$ and $f \notin (a)$. However, no power of g is divisible by a, as f and g are coprime so g has no f's in its factorization.

We can also prove this via the Chinese Remainder Theorem. Let $a = \prod_{i=1}^{n} f_i^{e_i}$. Then $R/(a) \cong \prod_{i=1}^{n} R/(f_i)^{e_i}$, which satisfies the condition in (c) iff $n=1$.

3) As R is a domain, the kernel of $R \to qf(R)$ is trivial, so we identify R with its image via $r \mapsto \frac{r}{1}$.

All localizations of R live in $qf(R)$, by UP. Indeed,

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, hook, bend right] & S^{-1}R \arrow[d, hook] \\ & qf(R) \end{tikzcd}$$

so we identify all localizations as in $qf(R)$.

Then $R \subseteq R_m$ $\forall m$, so $R \subseteq \bigcap_m R_m$

On the other hand, let $a \in \bigcap_m R_m$. We want to say its denominator must be a unit, as it's not in any max'l ideal, but we need more rigor as representations aren't unique. Also, R is not assumed to be a UFD, so we can't just take a reduced form.

So instead, consider $I = \{ a \in R \mid aq \in R \}$. This is the set of denominators of q. You can see clearly that it's an ideal of R.

Suppose $I \subseteq m$ a maximal ideal. If $q \in R_m$, $q = \frac{a}{b}$ w/ $b \notin m$. But $bq \in R$ so $b \in I \subseteq m$ $\times$.

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