Algebraic Geometry

study <u>varieties</u>, solutions to sets of polynomials

Rmk. we work over $\mathbb{C}$

all our pictures are in $\mathbb{R}$

Def. $\mathbb{C}^n$ is called "affine n-space $/\mathbb{C}$"
(often written $\mathbb{A}^n$, $\mathbb{A}^n_{\mathbb{C}}$, $\mathbb{A}^n(\mathbb{C})$)

In AG, the philosophy is to study spaces via functions on them.

for $\mathbb{C}^n$, these are $\mathbb{C}[x_1, \dots, x_n]$

Def. Let $S \subseteq \mathbb{C}[x_1, \dots, x_n]$, we define

$$V(S) = \{p \in \mathbb{C}^n \mid f(p) = 0 \ \forall f \in S\}$$

$$= \bigcap_{f \in S} \{p \in \mathbb{C}^n \mid f(p) = 0\}$$

such a set is called a <u>variety</u>, or an <u>algebraic subset</u> of $\mathbb{C}^n$.

Rmks. - $V(S) = V(I)$ for $I = (S)$

- $V(I+J) = V(I) \cap V(J)$
- $V(IJ) = V(I \cap J) = V(I) \cup V(J)$
- $V((0)) = \mathbb{C}^n$
- $V((1)) = \emptyset$

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Def. Let $Y \subseteq \mathbb{C}^n$. We let $I(Y) = \{ f \in \mathbb{C}[x_1, \dots, x_n] \mid f(y) = 0 \ \forall y \in Y \}$

Rmk. If $f^n \in I(Y)$ then $f \in I(Y)$. That is, $I(Y)$ is a radical ideal

Def. R a comm. ring, $I \subseteq R$ an ideal is called <u>radical</u> if $\forall f^n \in I$, $f \in I$.

We let $\sqrt{I} = \{ f \in R \mid \exists n \ f^n \in I \}$ called the radical of I

This is an ideal of R

e.g. $\sqrt{(0)} = \mathrm{nil}(R)$

Rmk. $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian, so any ideal I is f.g., so any variety is a finite intersection of "hypersurfaces", i.e. $V(f)$

So, fns on $\mathbb{C}^n$ are $\mathbb{C}[x_1, \dots, x_n]$

what are fns on a variety $Y \subseteq \mathbb{C}^n$?

We want these to be polynomials, but what is the precise ring?

$$\mathbb{C}[x_1, \dots, x_n] \longrightarrow \operatorname{Fun}(Y, \mathbb{C})$$
$$f \longmapsto f|_Y$$

has kernel $I(Y)$!

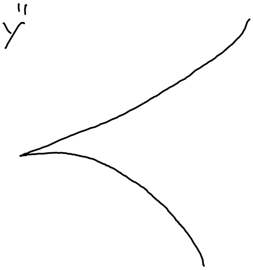
Def. For a variety $Y \subseteq \mathbb{C}^n$, let $\mathcal{O}(Y) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{I(Y)}$

That is, two fns are identified if they restrict to the same function on Y.

algebraic properties of $\mathcal{O}(Y)$ $\longleftrightarrow$ geometric properties of Y

e.g. $\{y^2 = x^3\} \subseteq \mathbb{C}^2$

Y ''

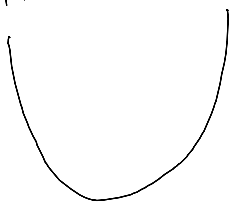


$$\mathcal{O}(Y) = \frac{\mathbb{C}[x, y]}{(y^2 - x^3)} \cong \mathbb{C}[t^2, t^3] \text{ via } \begin{array}{cc} x & y \\ \updownarrow & \updownarrow \\ t^2 & t^3 \end{array}$$

i.e. $t = \frac{y}{x}$, which is not defined on Y

$Y = V(y - x^2) \subseteq \mathbb{C}^2$

$\|$



$$\mathcal{O}(Y) = \frac{\mathbb{C}[x,y]}{(y-x^2)} \cong \mathbb{C}[t] = \mathcal{O}(\mathbb{C}^1)$$

$$x \longmapsto t$$
$$y \longmapsto t^2$$

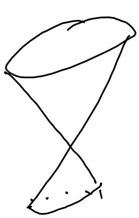
and indeed, $Y \longrightarrow \mathbb{C}$ is an "isomorphism"

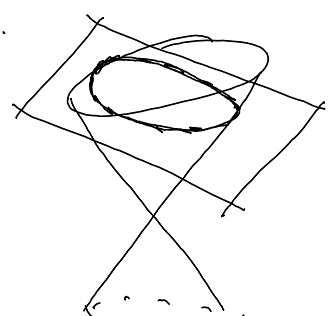
$$(x,y) \longmapsto x$$

$\cup$

$\downarrow$

$Y = V(z^2 - (x^2+y^2))$. $Z = V(z^2 - (x^2+y^2),\ x+y+z)$





Now, fix a variety Y. We can do the same thing we did with $\mathbb{C}^n$ replaced with Y.

Def. Let $S \subseteq \mathcal{O}(Y)$, $V(S) := \{p \in Y \mid f(p) = 0 \ \forall f \in S\}$
↳ alg. subset of Y

Then $V(S) = V((S)) = V(\sqrt{(S)})$ as before

Let $Z \subseteq Y$. $I(Z) := \{f \in \mathcal{O}(Y) \mid f(z) = 0 \ \forall z \in Z\}$

Big Theorem. Let Y be a variety.

$$\{\text{radical ideals of } \mathcal{O}(Y)\} \xleftrightarrow{\sim} \{\text{alg. subsets of } Y\}$$
$$J \longmapsto V(J)$$
$$I(Z) \longleftarrow\!\mapsto Z$$

s.t. - $V(I+J) = V(I) \cap V(J)$

- $V(IJ) = V(I \cap J) = V(I) \cup V(J)$

- order reversing

This restricts to

* $\{\text{maximal ideals of } \mathcal{O}(Y)\} \xleftrightarrow{\sim} \{\text{pts of } Y\}$

* Let $m_p = I(p)$, $p \in Y$. $\mathcal{O}(Y)/m_p \xrightarrow{\sim} \mathbb{C}$

$$f \longmapsto f(p)$$

Rmk, there is a subtle inconsistency to resolve here.

Suppose we had $Z \subseteq Y \subseteq \mathbb{C}^n$

$Y \subseteq \mathbb{C}^n$ algebraic

$Z \subseteq Y$ algebraic

Questions

- Is Z an algebraic subset of $\mathbb{C}^n$?

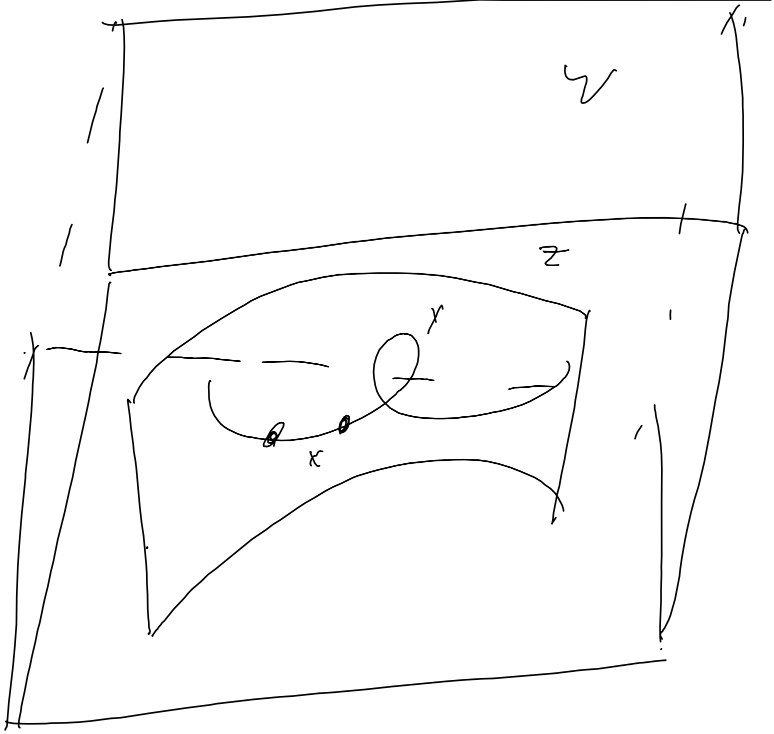
  i.e. is algebraicity transitive?

- What does $I(Z)$ mean? Is it an ideal of $\mathcal{O}(Y)$ or $\mathcal{O}(\mathbb{C}^n)$?

It doesn't matter, based on the correspondence principle!

$$Z \subseteq Y \iff I(Y) \subseteq I(Z)$$

$$\begin{array}{ccc}
\{\text{ideals of } \mathcal{O}(Y)\} & \cong & \{\text{ideals of } \mathcal{O}(\mathbb{C}^n) \text{ containing } I(Y)\} \\
\downarrow & & \downarrow \\
\{\text{alg. subsets of } Y\} & \cong & \{\text{alg. subsets of } \mathbb{C}^n \text{ contained in } Y\} \\
\{\text{maxl ideals of } \mathcal{O}(Y)\} & \cong & \{\text{maxl ideals of } \mathbb{C}^n \text{ containing } I(Y)\} \\
\wr\| & & \wr\| \\
\{\text{pts of } Y\} & \cong & \{\text{pts of } \mathbb{C}^n \text{ contained in } Y\}
\end{array}$$



3rd iso thm says $\dfrac{R/I}{J/I} = R/J$

Take again $Z \subseteq Y \subseteq \mathbb{C}^n$

$$\mathcal{O}(\mathbb{C}^n) \twoheadrightarrow \mathcal{O}(Y) \twoheadrightarrow \mathcal{O}(Z)$$

$$\mathcal{O}(Y) = \frac{\mathcal{O}(\mathbb{C}^n)}{I(Y)}$$

$$\mathcal{O}(Z) = \frac{\mathcal{O}(\mathbb{C}^n)}{I(Z)} = \frac{\mathcal{O}(\mathbb{C}^n)/I(Y)}{I(Z)/I(Y)} = \frac{\mathcal{O}(Y)}{I(Z)}$$

Morphisms of varieties

Let $X \subseteq \mathbb{C}^n$, $Y \subseteq \mathbb{C}^m$ varieties.

A map $X \longrightarrow Y$ is the restriction of a polynomial function $\mathbb{C}^n \longrightarrow \mathbb{C}^m$

This forms a category $\underline{\text{Var}}$

Thm. $\underline{\text{Var}}(X,Y) \cong \underline{\mathbb{C}\text{-Alg}}(\mathcal{O}(Y), \mathcal{O}(X))$

$\varphi \longmapsto (f \overset{\varphi^*}{\longmapsto} f \circ \varphi)$

Take e.g. $Y = \mathbb{C}^m$ itself.

To define a map $X \longrightarrow \mathbb{C}^m$ is to define m maps $X \longrightarrow \mathbb{C}$, i.e. m elements of $\mathcal{O}(Y)$, i.e. a map $\mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathcal{O}(Y)$

More generally, let $\mathcal{O}(Y) = \dfrac{\mathbb{C}[y_1, \ldots, y_m]}{I(Y)}$. The y_i are the coordinate functions on X. Given $\mathcal{O}(Y) \xrightarrow{\alpha} \mathcal{O}(X)$. Then $\alpha(y_i)$ are supposed to be "pullbacks" of coordinate fns along some map $X \longrightarrow Y$

So we define $X \longrightarrow Y$ via

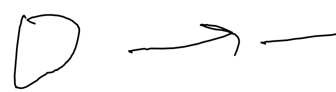
$$x \longmapsto \begin{pmatrix} \alpha(y_1)(x) \\ \vdots \\ \alpha(y_m)(x) \end{pmatrix}$$

e.g.,

$$\begin{aligned} \mathbb{C}[x,y] &\longrightarrow \mathbb{C}[x] \\ x &\longmapsto x \\ y &\longmapsto 0 \end{aligned}$$

$$\begin{aligned} \mathbb{C}^2 &\longleftarrow \mathbb{C}^1 \\ \begin{pmatrix} x \\ 0 \end{pmatrix} &\longleftarrow\!\shortmid\ x \end{aligned}$$

$$\begin{aligned} \mathbb{C}[x] &\longrightarrow \mathbb{C}[x,y] \\ x &\longmapsto x \end{aligned}$$



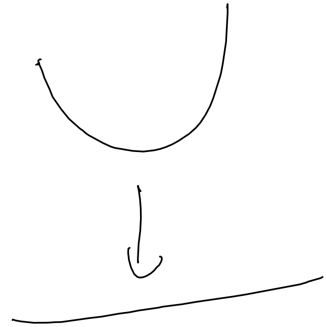
$$\begin{aligned} \mathbb{C} &\longleftarrow \mathbb{C}^2 \\ x &\longleftarrow\!\shortmid \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbb{C}[t] &\longrightarrow \mathbb{C}[x,y] \\ t &\longmapsto x+y \end{aligned}$$

$$\begin{aligned} \mathbb{C} &\longleftarrow \mathbb{C}^2 \\ x+y &\longleftarrow\!\shortmid \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

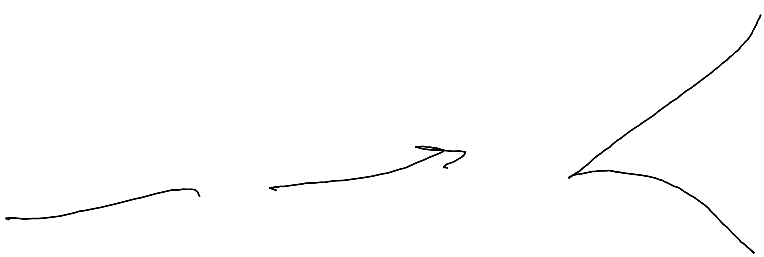
$$\mathbb{C}[x] \longrightarrow \frac{\mathbb{C}[x,y]}{(y-x^2)}$$

$$x \longmapsto x$$



$$x \longmapsto t^3$$
$$y \longmapsto t^2$$

$$\begin{tikzcd}
\frac{\mathbb{C}[x,y]}{(x^2-y^3)} \arrow[r] \arrow[dr, "\sim"'] & \mathbb{C}[t] \\
& \mathbb{C}[t^2,t^3] \arrow[u, hook]
\end{tikzcd}$$



$$t \longmapsto (t^3, t^2)$$