

# Admin

Hw 1 due 11:59 PM } does gradescope work?

Hw 2 due next Th

(++) is double the pts

do prob. 1 (other q's)

## Review of categories

Categories have objects "bricks"

Morphisms

"verbs"  $\xrightarrow{\text{idm. fib}}$

and Composition

e.g. Set, R-Mod, Group, Ab, Euclid, Euclid+, B4, ...

Functors are maps between categories "constructions"

$$F: \mathcal{C} \longrightarrow \mathcal{D}$$

$$\text{obj}(\mathcal{C}) \longrightarrow \text{obj}(\mathcal{D})$$

$$\mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$$

$$\text{s.t. } F(fg) = F(f) \circ F(g)$$

$$F(\text{id}_A) = \text{id}_{F(A)}$$

ex:  $F: \text{Ring} \longrightarrow \text{Group}$   
 $R \longmapsto R^\times$

- various "forgetful" functors

- Free:  $\text{Set} \longrightarrow \text{Group}$

- Euclid  $\longrightarrow \mathbb{R}\text{-Mod}$

$(V, q) \longmapsto \dim V$

$f \longmapsto df|_q$  the Jacobian matrix

-  $F: \underline{B} \mathcal{G} \longrightarrow \mathcal{C}$  (objects of

- an object  $A = F(*)$

-  $g \text{ map } \mathcal{G} \longrightarrow \mathcal{C}(A, A)$

$\parallel$   
 $\text{End}_{\mathcal{C}}(A)$

in fact, the map will give us  $\text{Aut}_{\mathcal{C}}(A)$

$\mathcal{G} \longrightarrow \text{Aut}_{\mathcal{C}}(A)$

$\mathcal{G}$  acting on  $A$

ex:  $\mathcal{G}$ -set,  $\mathcal{G}$ -rep,  $\mathcal{G}$ -space, etc.

-  $\mathcal{C}(A, -)$

-  $\mathcal{C}(-, A)$  (must define  $\mathcal{C}^{\text{op}}$ )

# Naturality

History of category theory w/ "natural equivalences"

e.g.  $V \in \mathbb{R}\text{-Vect}^{\text{fd}}$

Consider  $(-)^*$ :  $\mathbb{R}\text{-Vect}^{\text{fd}} \xrightarrow{\text{op}} \mathbb{R}\text{-Vect}^{\text{fd}}$   
 $(-)^{**}$ :  $\mathbb{R}\text{-Vect}^{\text{fd}} \rightarrow \mathbb{R}\text{-Vect}^{\text{fd}}$

we furthermore know  $V \xrightarrow{\sim} V^{**}$  "relating constructions"

by evaluation

so the identity functor and  $(-)^{**}$  are related

Further, for  $f: V \rightarrow W$ , we have

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow f & \Downarrow & \downarrow f \\ V & \xrightarrow{f} & W \end{array}$$

we say  $f$  and  $(-)^{**}$  are naturally isomorphic

"uniform in across all  $\mathbb{R}\text{-Vect}^{\text{fd}}$ "

in contrast,  $(-)^*$  is not due to the iso  $U \cong U^*$   
 coming from bases  
 fix iso  $U \xrightarrow{\varphi_U} U^*$  for all  $U$

$$\begin{array}{ccc}
 U & \xrightarrow{\varphi_U} & U^* \\
 \sim & & \sim \\
 f \downarrow & & \uparrow f^t \\
 W & \xrightarrow{\varphi_W} & W^*
 \end{array}$$

If this commutes,  $f$  must be surjective, as must  $f^t$   
 which is already surjective

further,  $\varphi_U = f^t \circ \varphi_W \circ f$  for all iso  $U \xrightarrow{\sim} W$   
 take e.g.  $U = W \cong \mathbb{R}^n$  Then for all  $A \in GL_n(\mathbb{R})$ ,  $\varphi_U = A^t \varphi_W A$   
hermitian

Def. Let  $\mathcal{C} \xrightarrow{F} \mathcal{D}$   
 $\eta$

$\eta$  natural transformation  $\eta: F \Rightarrow G$ ,

written  $\mathcal{C} \xrightarrow{F} \mathcal{D}$   
 $\eta$

is a collection of maps

$$\eta_A: F(A) \longrightarrow G(A)$$

for all objects  $A$  of  $\mathcal{C}$

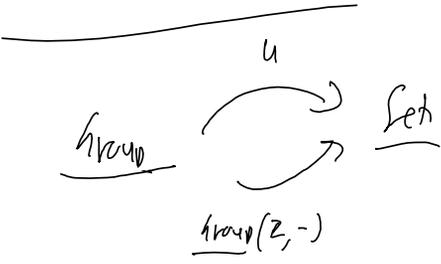
Let  $f: A \longrightarrow B$  in  $\mathcal{C}$ ,

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ F(f) \downarrow & \curvearrowright & \downarrow G(f) \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

It's a nat. iso if  $\eta_A$  are isos for all  $A$ .

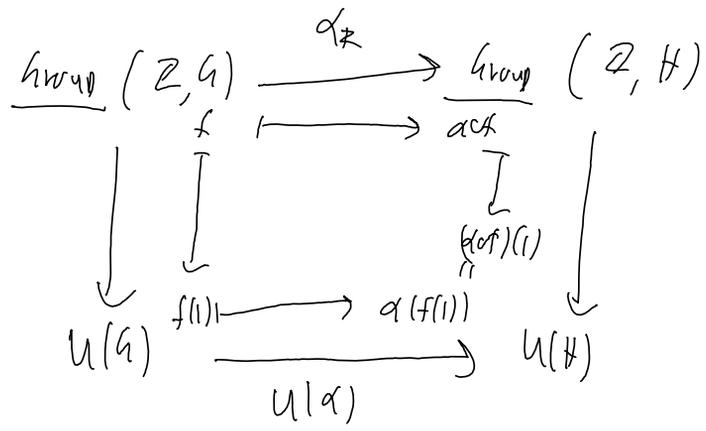
e.g.  $\text{id} \Rightarrow (-)^{**}$  is a nat iso

other examples



$$\begin{array}{ccc}
 \text{Group}(\mathbb{Z}, G) & \xrightarrow{\quad \sim \quad} & U(G) \\
 f \downarrow & & \downarrow f(i)
 \end{array}$$

$$d', G \longrightarrow H$$



Thus,  $\text{Group}(\mathbb{Z}, -) \cong U$

More generally,  $\text{Group}(F(S), -) \cong \text{Set}(S, U(-))$

$$\begin{array}{ccc}
 \text{Ring} & \xrightarrow{(-)^{\times}} & \text{Group} \\
 \downarrow & \searrow & \\
 \text{Ring}(\mathbb{Z}[x, x^{-1}], -) & & 
 \end{array}$$

$$\begin{array}{ccc}
 \text{Ring}(\mathbb{Z}[x, x^{-1}], R) & \xrightarrow{\sim} & R^{\times} \\
 f_1 \longrightarrow & & f(x)
 \end{array}$$

$$- \Omega = \{0, i^2\}$$

$$\begin{array}{ccc}
 \text{Set}(S, \Omega) & \xrightarrow{\sim} & \mathcal{P}(S) \\
 f_1 \longrightarrow & & f^{-1}(1)
 \end{array}$$

Let  $\alpha: S \rightarrow T$ , Then

$$\begin{array}{ccc}
 \text{Set}(T, \Omega) & \xrightarrow{\alpha^{\times}} & \text{Set}(S, \Omega) \\
 \downarrow \sim & & \downarrow \sim \\
 \mathcal{P}(T) & \longrightarrow & \mathcal{P}(S) \\
 \chi \longmapsto & & \alpha^{-1}(x)
 \end{array}$$

- Same w/  $\text{Set}$  replaced by  $\text{Top}$  & replaced by the set of all open subsets, and  $\Omega$  formalized via  $\{0, \{1\}, \Omega^{\times}$ .

Let  $N \trianglelefteq G$ ,  $\pi: G \rightarrow G/N$ ,

$$\begin{array}{ccc} \text{Group } (G/N, +) & \longrightarrow & \text{Group } (G, +) \\ f_1 & \longrightarrow & f_2 \circ \pi \end{array}$$

This is 1-1 as  $\pi$  is onto,

In fact, image is  $\{g: G \rightarrow H \mid g|_N \text{ constant}\}$

Now let  $N \trianglelefteq G$ ,  $K \trianglelefteq G$ ,  $N \subseteq K$ ,

$$\begin{aligned} \text{Group } \left( \frac{G/N}{K/N}, + \right) &\cong \{g|_N \rightarrow H \text{ constant on } K/N\} \\ &\cong \{g \xrightarrow{f} H \text{ constant on } N \text{ s.t.} \\ &\quad \tilde{g}: G/N \rightarrow H \text{ constant on } K/N\} \\ &\cong \{G \rightarrow H \text{ constant on } K\} \\ &\cong \text{Group } (G/K, +) \end{aligned}$$

So  $\text{Group } \left( \frac{G/N}{K/N}, - \right) \cong \text{Group } (G/K, -)$  as

functors  $\text{Group} \rightarrow \text{Set}$ , recall 3rd isom!