Admin

Hw 1 due 11:59 PM } does gradescope work?

Hw 2 due next Th

(**) is double the pts

do prob. 1 / other q's

Review of categories

Categories have objects "nouns", morphisms "verbs", ~~identity~~

and composition

e.g. Set, R-Mod, Group, Ab, Euclid, Euclid$_{\neq}$, B4, ...

Functors are maps between categories "constructions"

$$F: \mathcal{C} \longrightarrow \mathcal{D}$$

$$\mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D})$$

$$\mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$$

s.t. $F(f \circ g) = F(f) \circ F(g)$

$F(id_A) = id_{F(A)}$

e.g. - $F : \underline{\text{Ring}} \longrightarrow \underline{\text{Group}}$

$R \longmapsto R^\times$

- various "forgetful" functors

- Free: Set $\longrightarrow$ Group

- $\underline{\text{Euclid}_*} \longrightarrow \mathbb{R}\text{-}\underline{\text{Mat}}$

$(U, a) \longmapsto \dim U$

$f \longmapsto df|_a$ the Jacobian matrix

- $F : \underline{BG} \longrightarrow \mathscr{C}$ consists of

- an object $A = F(*)$

- a map $G \longrightarrow \mathscr{C}(A, A) = \mathrm{End}_{\mathscr{C}}(A)$

in fact, the image will all be isos, so

$G \longrightarrow \mathrm{Aut}_{\mathscr{C}}(A)$

G acting on A

e.g. G-Set, G-Rep, G-space, etc.

- $\mathscr{C}(A, -)$

- $\mathscr{C}(-, A)$ (must define $\mathscr{C}^{op}$)

Naturality

History of category theory w/ "natural equivalences"

e.g. $V \in \mathbb{R}\text{-}\underline{\text{Vect}}^{fd}$

Consider $(-)^*: \mathbb{R}\text{-}\underline{\text{vect}}^{fd\ op} \longrightarrow \mathbb{R}\text{-}\underline{\text{vect}}^{fd}$

$(-)^{**}: \mathbb{R}\text{-}\underline{\text{vect}}^{fd} \longrightarrow \mathbb{R}\text{-}\underline{\text{vect}}^{fd}$

we furthermore have

$$V \xrightarrow{\sim} V^{**}$$

"relating constructions"

by evaluation

so the identity functor and $(-)^{**}$ are related

Furthermore, for $f: V \to W$, we have

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & V^{**} \arrow[d] \\ W \arrow[r] & W^{**} \end{tikzcd}$$

we say id and $(-)^{**}$ are <u>naturally</u> isomorphic

"uniform iso across all $\mathbb{R}\text{-}\underline{\text{vect}}^{fd}$"

in contrast, $(-)^*$ is not due to the iso $V \cong V^*$

coming from bases

fix isos $V \xrightarrow{\varphi_V} V^*$ for all V

$$\begin{tikzcd}
V \arrow[r, "\varphi_V", "\sim"'] \arrow[d, "f"'] & V^* \\
W \arrow[r, "\sim", "\varphi_W"'] & W^* \arrow[u, "f^t"']
\end{tikzcd}$$

If this commutes, f must be an iso, as must f^t

which is already an issue

further, $\varphi_V = f^t \circ \varphi_W \circ f$ for all isos $V \xrightarrow{\sim} W$

take e.g. $V = W = \mathbb{R}^n$. Then for all $A \in GL_n \mathbb{R}$, $\varphi_V = A^t \varphi_V A$

<u>no shot</u>

Def. Let $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$

a natural transformation $\eta : F \Rightarrow G$,

written $\mathcal{C} \underset{G}{\overset{F}{\underset{\Downarrow \eta}{\rightrightarrows}}} \mathcal{D}$

is a collection of maps

$$\eta_A : F(A) \longrightarrow G(A)$$

for all objects A of $\mathcal{C}$

s.t. $\forall f : A \longrightarrow B$ in $\mathcal{C}$,

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowright"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

It's a nat. iso if η_A are isos for all A.

e.g. $id \Rightarrow (-)^{**}$ is a nat iso

other examples

$$\begin{tikzcd} \underline{\text{Group}} \arrow[r, bend left, "U"] \arrow[r, bend right, "\underline{\text{Group}}(\mathbb{Z},-)"'] & \underline{\text{Set}} \end{tikzcd}$$

$$\begin{aligned} \underline{\text{Group}}(\mathbb{Z}, G) &\xrightarrow{\sim} U(G) \\ f &\longmapsto f(1) \end{aligned}$$

$\alpha : G \longrightarrow H$

$$\begin{tikzcd} \underline{\text{Group}}(\mathbb{Z}, G) \arrow[r, "\alpha_*"] \arrow[d] & \underline{\text{Group}}(\mathbb{Z}, H) \arrow[d] \\ U(G) \arrow[r, "U(\alpha)"'] & U(H) \end{tikzcd}$$

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \alpha\circ f \arrow[d, mapsto] \\ f(1) \arrow[r, mapsto] & \alpha(f(1)) \end{tikzcd}$$

$(\alpha\circ f)(1) = \alpha(f(1))$

Thus, $\underline{\text{Group}}(\mathbb{Z}, -) \cong U$

More generally, $\underline{\text{Group}}(F(S), -) \cong \underline{\text{Set}}(S, U(-))$

- $$\begin{tikzcd} \underline{\mathrm{Ring}} \arrow[r, bend left, "(-)^\times"] \arrow[r, bend right, "\underline{\mathrm{Ring}}(\mathbb{Z}[x,x^{-1}],\, -)"'] & \underline{\mathrm{Group}} \end{tikzcd}$$

$$\underline{\mathrm{Ring}}(\mathbb{Z}[x,x^{-1}],\, R) \xrightarrow{\sim} R^\times$$
$$f \longmapsto f(x)$$

- $\Omega = \{0,1\}$

$$\underline{\mathrm{Set}}(S, \Omega) \xrightarrow{\sim} \mathcal{P}(S)$$
$$f \longmapsto f^{-1}[1]$$

Let $\alpha : S \to T$. Then

$$\begin{tikzcd} \underline{\mathrm{Set}}(T, \Omega) \arrow[r, "\alpha^*"] \arrow[d, "\sim"'] & \underline{\mathrm{Set}}(S, \Omega) \arrow[d, "\sim"] \\ \mathcal{P}(T) \arrow[r] & \mathcal{P}(S) \end{tikzcd}$$
$$X \longmapsto \alpha^{-1}[X]$$

- Same w/ $\underline{\mathrm{Set}}$ replaced by $\underline{\mathrm{Top}}$, $\mathcal{P}$ replaced by the set of open subsets, and Ω topologized via $\{\emptyset, \{1\}, \Omega\}$.

- Let $N \trianglelefteq G$, $\pi : G \longrightarrow G/N$.

$$\underline{\text{Group}}(G/N, H) \longrightarrow \underline{\text{Group}}(G, H)$$
$$f \longmapsto f \circ \pi$$

This is 1-1 as π is onto.

Its image is $\{ g : G \to H \mid g|_N \text{ constant} \}$

Now let $N \trianglelefteq G$, $K \trianglelefteq G$, $N \subseteq K$.

$$\underline{\text{Group}}\left(\frac{G/N}{K/N}, H\right) \cong \{ G/N \to H \text{ constant on } K/N \}$$

$$\cong \{ G \xrightarrow{g} H \text{ constant on } N \text{ s.t. } \bar{g} : G/N \to H \text{ constant on } K/N \}$$

$$\cong \{ G \to H \text{ constant on } K \}$$

$$\cong \underline{\text{Group}}(G/K, H)$$

So $\underline{\text{Group}}\left(\frac{G/N}{K/N}, -\right) \cong \underline{\text{Group}}(G/K, -)$ as functors $\underline{\text{Grp}} \longrightarrow \underline{\text{Set}}$. Recall 3rd iso!