Intro

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OH: This week, Th 4PM and F 12PM

later weeks, based on poll

office: MS 3919

HW: Thursdays @ 11:59PM (but I'm lenient in general)

only (*) graded for correctness

within 1 week of me publishing grades for HW, you may resubmit and I'll regrade

Section will typically be me first answering questions, such as on the homework, and then I'll present some new content I think is useful or interesting.

Categories

Reference; Riehl's "Category Theory in Context"
(free on her website)

- Category theory affords a unified language across various mathematical fields
- It provides a general, abstract framework to find surprising patterns in previously disparate proofs and constructions
- It emphasizes relationships between objects, rather than studying objects individually.

Categories will arise in nature as natural domains to discuss particular classes of objects.

Def. A category $\mathcal{C}$ consists of the following data:
- a collection of objects $\mathrm{Obj}(\mathcal{C})$ "nouns"
- For each pair $A, B \in \mathrm{Obj}(\mathcal{C})$, a set of morphisms $\mathrm{Hom}(A,B)$, $\mathrm{Mor}(A,B)$, $\mathcal{C}(A,B)$, $\mathrm{Hom}_{\mathcal{C}}(A,B)$. We sometimes write
- For $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a composition $\circ : \mathrm{Hom}(B,C) \times \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,C)$
$(g, f) \longmapsto g \circ f$

(we write $f: A \to B$ or $A \xrightarrow{f} B$ for $f \in \mathcal{C}(A, B)$)

- a morphism $id_A : A \to A$ for all objects A.

these data are subject to the following axioms.

- For $f: A \to B$, $f \circ id_A = f$ and $id_B \circ f = f$
- For $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, $h \circ (g \circ f) = (h \circ g) \circ f$

Examples.

- $\underline{Set} = \begin{cases} obj(\underline{Set}) = \text{collection of all sets} \\ \underline{Set}(A, B) = \text{functions } A \to B \\ \circ \text{ is the usual composition} \\ id_A \text{ is defined as } id_A(a) = a \end{cases}$

- $\underline{Group} = \begin{cases} obj(\underline{Group}) = \text{collection of all groups} \\ \underline{Group}(A, B) = \text{group homomorphisms } A \to B \\ \circ \\ id_A \end{cases}$ as before

- $\underline{Ring} = \begin{cases} obj = \text{rings} \\ \underline{Ring}(A, B) = \text{ring homomorphisms } A \to B \end{cases}$

- $\underline{CRing} = \begin{cases} obj = \text{commutative rings} \\ \underline{CRing}(A, B) = \underline{Ring}(A, B) \end{cases}$

etc. for $\underline{Ab}$, $\underline{Field}$, $\underline{Monoid}$, $\underline{Semigroup}$, $\underline{k\text{-Vect}}$, ...

abelian groups

There are non algebraic categories too

- $\underline{Top} = \begin{cases} \text{obj} = \text{topological spaces} \\ \underline{Top}(A,B) = \text{continuous maps } A \to B \end{cases}$

- $\underline{SmMan} = \begin{cases} \text{obj} = \text{smooth manifolds} \\ \underline{SmMan}(A,B) = \text{smooth maps } A \to B \end{cases}$

- $\underline{Euclid} = \begin{cases} \text{obj} = \text{open subsets of some } \mathbb{R}^n \\ \text{morphisms} = \text{smooth maps} \end{cases}$

etc. for complex manifolds, algebraic varieties...

We can add conditions

$\underline{Set}_* = \begin{cases} \text{obj} = \text{pairs } (A,a) \text{ of a set } A \text{ and an element } a \in A \\ \underline{Set}_*((A,a),(B,b)) = \{f : A \to B \mid f(a) = b\} \end{cases}$

"category of pointed sets"

same for $\underline{Top}_*$, $\underline{SmMan}_*$, $\underline{Euclid}_*$, etc.

All the above are basically just $\underline{Set}$ with extra conditions/data. We can go weirder.

- Let R be a ring. Define $R\text{-}\underline{Mat} = \begin{cases} \text{obj}(R\text{-}\underline{Mat}) = \mathbb{N} \\ R\text{-}\underline{Mat}(n,m) = M_{m \times n}(R) \\ id_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \\ \circ = \text{matrix multiplication} \end{cases}$

- $\underline{hTop}$ = $\begin{cases} \text{obj} = \text{top. spaces} \\ hTop(A,B) = Top(A,B)/\text{homotopy} \end{cases}$

- Let X be a topological space. Define the "fundamental groupoid" of X to be the category

$$\Pi_1(X) = \begin{cases} \text{obj}(\Pi_1(X)) = X \\ \text{Hom}(x,y) = \text{htpy classes of paths } x \to y \text{ in } X \\ id_x = \text{constant path at } x \\ \circ = \text{concatenation} \end{cases}$$

- Let $\mathbb{P} = (P, \leq)$ be a poset (partially ordered set), i.e. $\leq$ is a reflexive, transitive, and antisymmetric relation on P
$(x \leq y \leq x \Rightarrow x = y)$

we associate to $(P, \leq)$ a category

$$\underline{\mathbb{P}} = \begin{cases} - \text{obj}(\underline{\mathbb{P}}) = P \\ - \text{For } x, y \in P, \text{ we let } \underline{\mathbb{P}}(x,y) \text{ be a singleton if } x \leq y \text{ and } \phi \text{ if not} \\ - id_x \text{ is the unique element of } \underline{\mathbb{P}}(x,x) \text{ as } x \leq x \text{ by reflexivity} \\ - \text{Composition } \underline{\mathbb{P}}(y,z) \times \underline{\mathbb{P}}(x,y) \longrightarrow \underline{\mathbb{P}}(x,z) \\ \text{exists as the former two Hom sets are nonempty if } x \leq y \text{ and } y \leq z, \text{ whence } x \leq z \text{ by} \\ \text{transitivity and hence } \underline{\mathbb{P}}(x,z) \text{ is a singleton} \end{cases}$$

- Let G be a group.

$$\underline{BG} = \begin{cases} - \operatorname{obj}(BG) \text{ is a singleton, } \{*\} \\ - \underline{BG}(*,*) = G \\ - id_* = e_G \\ - \circ \text{ is the group multiplication} \end{cases}$$

$$- \underline{\text{Measure}} = \begin{cases} \operatorname{obj} = \text{measure spaces} \\ \underline{\text{Measure}}(A,B) = \dfrac{\{\text{measurable fns } A \to B\}}{f \sim g \text{ if } \{f \neq g\} \text{ has measure } 0} \end{cases}$$

We often write "let $f: A \to B$ in $\mathcal{C}$" to save room. e.g. we don't write "let A, B be rings and $f: A \to B$ a ring homomorphism"

First Principles

Def. Let $\mathcal{C}$ be a category. $\mathcal{C}^{op} = \begin{cases} \operatorname{obj}(\mathcal{C}^{op}) = \operatorname{obj}(\mathcal{C}) \\ \mathcal{C}^{op}(A,B) = \mathcal{C}(B,A) \end{cases}$ "flip the arrows around"

Def. Let $\mathcal{C}$ be a category and $f: A \to B$ in $\mathcal{C}$.

- f is a monomorphism if for all $g, h: T \to A$ in $\mathcal{C}$, we have $f \circ g = f \circ h \Rightarrow g = h$
- f is an epimorphism if for all $g, h: B \to T$ in $\mathcal{C}$, we have $g \circ f = h \circ f \Rightarrow g = h$, i.e. f is monic in $\mathcal{C}^{op}$
- f is an isomorphism if $\exists g: B \to A$ in $\mathcal{C}$ s.t. $f \circ g = id_B$ and $g \circ f = id_A$. This uniquely characterizes g, and we write $g = f^{-1}$.

- An iso $A \xrightarrow{f} A$ is called an automorphism of A. we write $\mathrm{Aut}(A)$ to collect these. They form a group under $\circ$.

Examples. - In Set

monomorphism $\Leftrightarrow$ injective

epimorphism $\Leftrightarrow$ surjective

isomorphism $\Leftrightarrow$ bijective

Rmk. injectivity and surjectivity are "dual"!

- In Group, the same goes as in Set. But it's hard to prove epic $\Leftrightarrow$ onto!

- In Ring

monic $\Leftrightarrow$ injective

epic $\Leftrightarrow$ surjective is FALSE

e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic

iso $\Leftrightarrow$ bijective still true

- In Top,

monic $\Leftrightarrow$ 1 1

epic $\Leftrightarrow$ onto

iso $\Leftrightarrow$ homeomorphism

- In the category of Hausdorff spaces, epic $\Leftrightarrow$ dense image

- In hTop, isos are htpy equivalences

- In $\mathbb{P}$, only the identity is an iso by antisymmetry

- In $\Pi_1(X)$, all morphisms are iso

- In BG, all morphisms are iso

we call these groupoids

Functors

We now, adhering to our philosophy, define maps between categories called functors. These often arise as constructions from one type of object to another.

Def. Let $\mathcal{C}$, $\mathcal{D}$ be categories. A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ consists of the following data:

- A function $\mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D})$
- functions $\mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$ for all $A, B \in \mathrm{Obj}(\mathcal{C})$

subject to the following axioms, called "functoriality"

- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$
- $F(f \circ g) = F(f) \circ F(g)$

Rmk. We can define $\underline{\mathrm{Cat}} = \begin{cases} \mathrm{Obj} = \text{categories} \\ \text{morphisms} = \text{functors}, \end{cases}$ but you'll scare the set theorists.

Examples

- There are "forgetful" functors taking a category with highly structured objects to one with less structured objects. For instance,

$$\begin{aligned} U : \underline{\text{Group}} &\longrightarrow \underline{\text{Set}} \\ (G, \cdot, e) &\longmapsto \text{its underlying set} \\ f &\longmapsto f \end{aligned}$$

$$\begin{aligned} U : \underline{\text{Ring}} &\longrightarrow \underline{\text{Ab}} \\ (R, +, \cdot, 0, 1) &\longmapsto (R, +, 0) \\ f &\longmapsto f \end{aligned}$$

$$\begin{aligned} U : \underline{\text{Ab}} &\longrightarrow \underline{\text{Group}} \\ (A, +, 0) &\longmapsto (A, +, 0) \\ f &\longmapsto f \end{aligned}$$

etc.

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$$\begin{tikzcd} \underline{\text{Ring}} \arrow[r] & \underline{\text{Group}} \\ \underline{\text{CRing}} \arrow[u, hook] \arrow[r] & \underline{\text{Ab}} \arrow[u, hook] \end{tikzcd}$$

via $R \longmapsto R^\times$

Lemma. If $f: A \to B$ is an isomorphism then $F(f)$ is an isomorphism

pf. $id_{F(A)} = F(id_A) = F(f^{-1} \circ f) = F(f^{-1}) \circ F(f)$

$id_{F(B)} = F(id_B) = F(f \circ f^{-1}) = F(f) \circ F(f^{-1})$

so in fact, $F(f^{-1}) = F(f)^{-1}$

as such, $R \cong S \Rightarrow R^\times \cong S^\times$.

- $\underline{Top_*} \longrightarrow \underline{Group}$
  $(X, x_0) \longmapsto \pi_1(X, x_0)$
  $f \longmapsto f_*$

  , In fact,
$$\begin{tikzcd} \underline{Top_*} \arrow[r] \arrow[d] & \underline{Group} \\ hTop_* \arrow[ur] & \end{tikzcd}$$

- $\underline{Top} \longrightarrow \underline{Groupoid}$
  $X \longmapsto \pi_1(X)$

- $F: \underline{Set} \longrightarrow \underline{Group}$ via
  $S \longmapsto F(S)$ the free group on S
  $S \xrightarrow{f} T \longmapsto F(S) \longrightarrow F(T)$
  via $\prod s_i \mapsto \prod f(s_i)$

- similarly, $F: \underline{Set} \longrightarrow \underline{Ab}$, $F: \underline{Set} \longrightarrow k\text{-}\underline{Vect}$

- $\underline{Euclid_*} \longrightarrow \mathbb{R}\text{-}\underline{Mat}$
  $(U, a) \longmapsto \dim U$
  $(U, a) \xrightarrow{f} (V, b) \longmapsto df|_a$, the Jacobian matrix. Functoriality is the chain rule!

- Let $\mathcal{C}$ be a category and G a group.

A functor $F : \underline{BG} \longrightarrow \mathcal{C}$ consists of

- a choice of object $A \in \mathcal{C}$ which we take as $F(*)$
- A group homomorphism $G \longrightarrow Aut(A)$

so this is an action of G on A

e.g.
- $\mathcal{C} = \underline{Set}$ recovers G-sets
- $\mathcal{C} = k\text{-}\underline{Vect}$ recovers representations of G
- $\mathcal{C} = \underline{Top}$ recovers continuous actions

- Let $\mathcal{C}$ be a category and $\mathbb{P} = (P, \leq)$ a poset.

A functor $\underline{\mathbb{P}} \xrightarrow{F} \mathcal{C}$ consists of

- objects $F(x)$ for all $x \in P$
- morphisms $f_{xy} : F(x) \longrightarrow F(y)$ for all $x \leq y$ s.t.

$f_{yz} \circ f_{xy} = f_{xz}$

e.g. $\underline{\mathbb{P}} = (\mathbb{N}, \leq)$, then this is a sequence of morphisms and objects

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots$$

- Let $\mathscr{C}$ be a category and $A \in \mathscr{C}$. We have the covariant hom functor

$$\begin{aligned} \mathscr{C}(A,-) : \mathscr{C} &\longrightarrow \text{Set} \\ B &\longmapsto \mathscr{C}(A,B) \\ B \xrightarrow{f} C &\longmapsto \left(\begin{aligned} \mathscr{C}(A,B) &\xrightarrow{\mathscr{C}(A,f) \text{ or } f_*} \mathscr{C}(A,C) \\ g &\longmapsto f \circ g \end{aligned} \right) \end{aligned}$$

and the contravariant hom functor

$$\begin{aligned} \mathscr{C}(-,A) : \mathscr{C}^{op} &\longrightarrow \text{Set} \\ B &\longmapsto \mathscr{C}(B,A) \end{aligned}$$

Let $f \in \mathscr{C}^{op}(C,B) = \mathscr{C}(B,C)$

Then $\mathscr{C}(f,A)$, or f^* is defined as

$$\begin{aligned} \mathscr{C}(C,A) &\longrightarrow \mathscr{C}(B,A) \\ g &\longmapsto g \circ f \end{aligned}$$

Rmk. f is monic $\Leftrightarrow f_*$ is injective
f is epic $\Leftrightarrow f^*$ is injective

A fundamental result in category theory called the Yoneda lemma says that all the "data" of A is "contained" in either its covariant or contravariant hom functor, i.e. by its "universal property"

I'll defer the formalism, and instead I'll "compute" some Hom functors.

- $$\begin{aligned} \underline{\mathrm{Group}}(\mathbb{Z}, G) &\xrightarrow{\sim} U(G) \\ f &\longmapsto f(1) \end{aligned}$$

Let $\alpha : G \to H$. Then

$$\begin{tikzcd}
\underline{\mathrm{Group}}(\mathbb{Z}, G) \arrow[r, "\alpha_*"] \arrow[d, "\sim"'] & \underline{\mathrm{Group}}(\mathbb{Z}, H) \arrow[d, "\sim"] \\
U(G) \arrow[r, "\alpha"'] & U(H)
\end{tikzcd}$$

$$f \longmapsto \alpha \circ f$$
$$(\alpha\circ f)(1) = \alpha(f(1))$$
$$f(1) \longmapsto \alpha(f(1))$$

we say this diagram "commutes"

So $\underline{\mathrm{Group}}(\mathbb{Z}, -)$ is "isomorphic" to $U : \underline{\mathrm{Group}} \to \underline{\mathrm{Set}}$

- More generally, $$\begin{aligned} \underline{\mathrm{Group}}(F(S), G) &\xrightarrow{\sim} \underline{\mathrm{Set}}(S, U(G)) \\ f &\longmapsto f|_S \end{aligned}$$

- Similarly, $\underline{\mathrm{Ab}}(F(S), A) \xrightarrow{\sim} \underline{\mathrm{Set}}(S, U(A))$

- $$\begin{aligned} \underline{\mathrm{Ring}}(\mathbb{Z}[x, x^{-1}], R) &\xrightarrow{\sim} R^\times \\ f &\longmapsto f(x) \end{aligned}$$

where $\mathbb{Z}[x, x^{-1}] = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \;\middle|\; \begin{array}{l} \text{all but finitely many } a_n = 0, \\ a_n \in \mathbb{Z} \end{array} \right\}$

Let $f : R \to S$. Then

$$\begin{tikzcd}
\underline{\mathrm{Ring}}(\mathbb{Z}[x, x^{-1}], R) \arrow[r, "f_*"] \arrow[d, "\sim"'] & \underline{\mathrm{Ring}}(\mathbb{Z}[x, x^{-1}], S) \arrow[d, "\sim"] \\
R^\times \arrow[r, "f|_{R^\times}"'] \arrow[ur, phantom, "\circlearrowleft"] & S^\times
\end{tikzcd}$$

- Let $\Omega = \{0,1\}$. Then

$$\begin{array}{ccc} \underline{\mathrm{Set}}(S, \Omega) & \xrightarrow{\sim} & P(S) \\ f & \longmapsto & f^{-1}(1) \end{array}$$

If $\alpha : S \to T$ then

$$\begin{tikzcd} \underline{\mathrm{Set}}(T, \Omega) \arrow[r, "\alpha^*"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowright"] & \underline{\mathrm{Set}}(S, \Omega) \arrow[d, "\sim"] \\ P(T) \arrow[r] & P(S) \end{tikzcd}$$

$$A \longmapsto \alpha^{-1}(A)$$

- Let $\Omega = \{0,1\}$ with the topology $\{\phi, \{1\}, \Omega\}$.

Then
$$\begin{array}{ccc} \underline{\mathrm{Top}}(X, \Omega) & \xrightarrow{\sim} & \mathrm{Open}(X) \\ f & \longmapsto & f^{-1}(1) \end{array}$$

where $\mathrm{Open}(X) = \{ U \subseteq X \text{ open} \}$.

Let $\alpha : X \to Y$ in $\underline{\mathrm{Top}}$. Then

$$\begin{tikzcd} \underline{\mathrm{Top}}(Y, \Omega) \arrow[r] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowright"] & \underline{\mathrm{Top}}(X, \Omega) \arrow[d, "\sim"] \\ \mathrm{Open}(Y) \arrow[r] & \mathrm{Open}(X) \end{tikzcd}$$

$$U \longmapsto \alpha^{-1}(U)$$

- Let $N \trianglelefteq G$ and $\pi : G \longrightarrow G/N$

$$\underline{\text{Group}}(G/N, H) \longrightarrow \underline{\text{Group}}(G, H)$$
$$f \longmapsto f \circ \pi$$

This is injective as π is onto.

In fact,

$$\underline{\text{Group}}(G/N, H) \xrightarrow{\sim} \{ g : G \to H \mid g|_N \text{ constant} \}$$
$$f \longmapsto f \circ \pi$$

Now consider $N \trianglelefteq G$, $K \trianglelefteq G$, and $N \leq K$.

Then

$$\underline{\text{Group}}\left(\frac{G/N}{K/N}, H\right) \cong \{ G/N \to H \text{ which are constant on } K/N \}$$

$$\cong \{ g : G \to H \text{ which are constant on } N \text{ and s.t. } \bar{g} : G/N \to H \text{ is constant on } K/N \}$$

$$\cong \{ g : G \to H \mid g \text{ constant on } K \}$$

$$\cong \underline{\text{Group}}(G/K, H)$$

So $\underline{\text{Group}}\left(\frac{G/N}{K/N}, H\right) \cong \underline{\text{Group}}(G/K, H)$, as expected by the 3rd iso thm!