# Symmetric Matrices / Spectral theorem

Statement. Let $W$ $n \times n$ matrix which is symmetric

$(W = W^T,\ w_{ij} = w_{ji})$

$W$ is "orthonormally diagonalizable"

i.e. there is some $\vec{v}_1, \dots, \vec{v}_n$ $\underline{\text{ONB}}$ of $\mathbb{R}^n$
orthonormal basis

of eigenvectors of $W$.

so in this basis, $W$ diagonal.

there is some orthogonal matrix $O$ so that

$O^{-1} W O$ is diagonal.

recall that. $O$ orthogonal $\iff O^{-1} = O^T$
$\iff$ col's being an ONB

Converse: $O^T W O = D$

$$(O^T W O)^T = D^T = D$$

$$\|$$

$$O^T W^T O$$

$$O^T W^T O = O^T W O \Rightarrow W = W^T$$

????

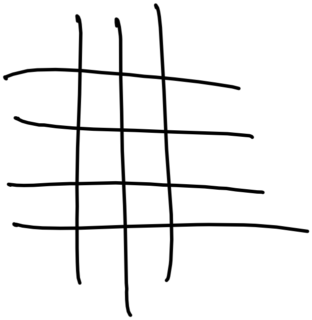
symmetry???

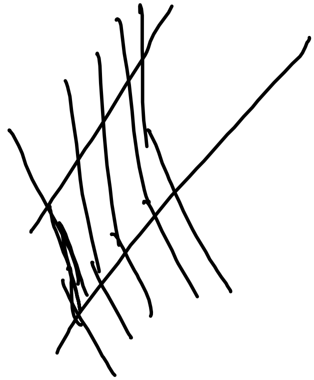
??  why??

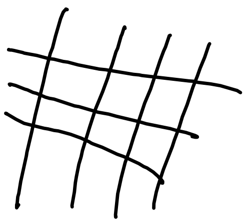
huh?

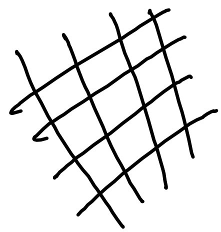
$$O^T W O = D$$

$$W = \underbrace{O}_{\text{undo rigid transform}}\ \underbrace{D}_{\text{scale along axes}}\ \underbrace{O^T}_{\text{rigid transformation}}$$







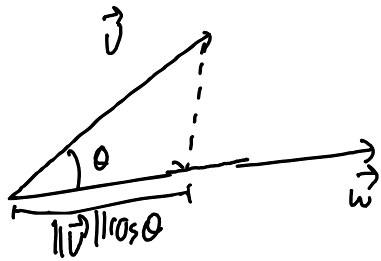


# Transpose, dot product, geometry

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i$$

$$= \vec{v}^T \vec{w}$$ transpose

$$= \|\vec{v}\| \, \|\vec{w}\| \cos(\theta)$$ geometry




$$(A^T)_{ij} = A_{ji}$$

$$(A\vec{v}) \cdot \vec{w} = (A\vec{v})^T \vec{w}$$
$$= \vec{v}^T A^T \vec{w}$$
$$= \vec{v}^T (A^T \vec{w})$$
$$= \vec{v} \cdot (A^T \vec{w})$$

$$\boxed{(A\vec{v})\cdot\vec{w} = \vec{v}\cdot(A^T\vec{w}).}$$

$$A = A^T$$

$$\boxed{(A\vec{v})\cdot\vec{w} = \vec{v}\cdot(A\vec{w})} \quad *$$

Application. $\vec{v}$ eigenvector of $A$, $A$ · symmetric

$$A\vec{v} = \lambda\vec{v}$$

$\vec{w}$ is perpendicular to $\vec{v}$

$\vec{w}$ in $\operatorname{span}(\vec{v})^\perp$

$$\vec{w} \perp \vec{v}$$

$$\boxed{\vec{w}\cdot\vec{v} = 0}$$

$$\begin{aligned}(A\vec{w})\cdot\vec{v} &= \vec{w}\cdot(A\vec{v}) \\ &= \vec{w}\cdot(\lambda\vec{v}) \\ &= \lambda\, \cancelto{0}{\vec{w}\cdot\vec{v}} \\ &= 0\end{aligned}$$

$\vec{w}$ is in $\text{span}(\vec{v})^\perp$ $\quad\quad$ $\vec{v} \perp \vec{w}$

$$\Downarrow \quad\quad\quad\quad \Downarrow$$

$A\vec{w}$ is in $\text{span}(\vec{v})^\perp$ $\quad$ $\vec{v} \perp (A\vec{w})$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\quad \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\quad A\vec{v} = \vec{v}$,

$\vec{v} \perp \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{w}$

$A\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \not\perp \vec{v}$

$\text{span}(\vec{v})^\perp$ $A$-invariant subspace

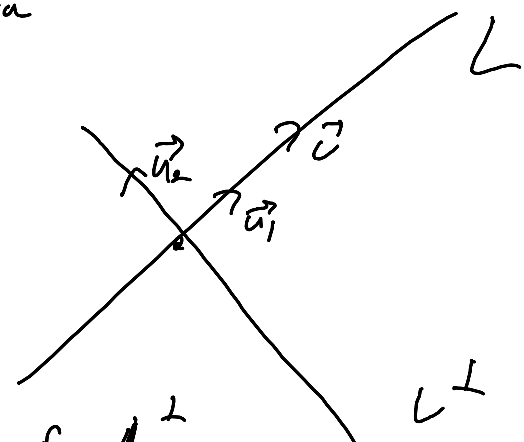
if $A$ symmetric

(normal matrix)

$\vec{v} \neq \vec{0}$

$\vec{u}_1 = \dfrac{\vec{v}}{\|\vec{v}\|}$ unit vector

$\text{span}(\vec{v}) = \text{span}(\vec{u}_1) = L$




$\vec{u}_2, \dots, \vec{u}_n$ orthonormal basis for $L^{\perp}$

$\vec{u}_1$ in $L$

$\vec{u}_2, \dots, \vec{u}_n$ in $L^{\perp}$

$\vec{u}_1, \dots, \vec{u}_n$ ONB $\mathbb{R}^n$

what is the matrix for $A$ in this basis?

$$A\vec{u}_1 = \lambda \vec{u}_1 = \underline{\lambda}\vec{u}_1 + \underline{0}\vec{u}_2 + \dots + \underline{0}\vec{u}_n \qquad \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$A\vec{u_2}$ in $L^\perp$

$\vec{u_2}$ in $L^\perp \underset{\text{symmetry of } A}{\Rightarrow} A\vec{u_2}$ in $L^\perp$

$$A\vec{u_2} = r_{22}\vec{u_2} + r_{23}\vec{u_3} + \dots + r_{2n}\vec{u_n}$$

$r_{2i}$ in $\mathbb{R}$, scalars

no $\vec{u_1}$-component

$$A\vec{u_2} \quad \begin{pmatrix} 0 \\ r_{22} \\ r_{23} \\ \vdots \\ r_{2n} \end{pmatrix}$$

$$\begin{array}{l} A\vec{u_i} \\ i \geq 2 \end{array} \quad \begin{pmatrix} 0 \\ r_{i2} \\ \vdots \\ r_{in} \end{pmatrix}$$

A in this new basis $\rightarrow$

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{?\ ?\ ?} & \\ 0 & & & \end{pmatrix} \leftarrow (n-1)\times(n-1) \text{ smaller}$$

Reduced to a smaller case, $n-1 < n$

recursively get a diagonal matrix

Issue: I assumed $M$ had an real eigenvector

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ no real eigenvectors

This is true, sym matrices have real eigenvectors and eigenvalues

# Why care?

Diagonal matrices are simpler

many matrices are similar to a diagonal one

If our problem is basis-independent, often we can work mostly with diagonal matrices

$$S^{-1}AS = D$$

$$(S^{-1}AS)^n = \underbrace{D^n}$$ ← easy to compute

$$\shortparallel$$

$$\underbrace{S^{-1}A^nS}_{n}$$

$$A^n = \underbrace{S}\underbrace{D^n}\underbrace{S^{-1}}$$

sometimes things aren't basis indep

A being symmetric is <u>not</u> basis indep

$A$ satisfies ____

$\Updownarrow$ iff

$S^{-1}AS$ satisfies ____

Then ____ is "basis independent"

symmetric is not basis indep

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Start w/ sym, $A = \begin{pmatrix} 1 & 3 \\ 2 & 50 \end{pmatrix}$

$S$ not orthog, $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$S^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$S^{-1}AS = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 50 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -48 \\ 2 & 52 \end{pmatrix}$$

not symmetric

Symmetric

indep of

ONB change

trick sym

applied a non-orthogonal basis change

Some properties are preserved under ONB basis change.

ONB -independence

A satisfies $\underline{\qquad}$

$\Updownarrow$

$O^T A O$ satisfies $\underline{\qquad}$

| indep of basis change | indep of ONB change |
|---|---|
| - determinant $\det(S^{-1}AS) = \det(A)$ | |
| - eigenvalues | |
| - char poly | |
| - rank | |
| - nullity | |
| - invertibility | |
| - diagonalizability | |

$\Longrightarrow$

$\not\Longleftarrow$ orthogonality

$\not\Longleftarrow$ symmetry
orthonormal diagonalizability

"anything involving transposes or angles/distance"

$\not\Longleftarrow$ $W\|\vec{v}\| = 2\|\vec{v}\|$ $\forall\vec{v}$

$W$ does this $\Longleftrightarrow$ $O^TWO$ does this
$O$ orthogonal

$A^{-1} \stackrel{?}{=} A^2$

$\not\Longleftarrow$ $W^T = W^2 \Longleftrightarrow (O^TWO)^T = (O^TWO)^2$

# Midterm problems

T/F. $A = A^{-1} \overset{??}{\Longrightarrow} A$ orthogonal?

involves a transpose

(F). $\boxed{A=A^{-1}}$ indep. of the choice of basis

$(S^{-1}AS)^{-1} = S^{-1}AS$

A orthog indep of the choice of ONB

$\underline{O} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$ orthogonal, $O^{-1}=O$

$S=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ invertible, not orthogonal

$S^{-1} = \begin{pmatrix}1&-1\\0&1\end{pmatrix} \neq S^T$

$A = \underline{S^{-1}}O\underline{S} = \begin{pmatrix}-1&-2\\0&1\end{pmatrix}$ counterexample

$A^{-1}=A$. <u>not</u> orthogonal. $A^T=A^{-1}=A$

Marg l': Change of basis messes up transpose

ONB change nice to transpose

$\boxed{\text{Computation}}$ $W$ symmetric

We have some ONB of eigenvectors

diagonalization

G-S on each eigenspace or $\ker(A - \lambda I)$

$$W = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

1. det computation

char poly: $-\lambda^3 + 3\lambda + 2$

2. find roots of poly
   - Cardano's method
   - rational root thm
   - long division

2.i) factor char poly

$$= -(\lambda - 2)(\lambda + 1)^2$$

$\lambda = 2$ $\quad$ $\lambda = -1$

a.m $=1$ $\quad$ a.m $= 2$

$\lambda = -1$, $\ker(W + I)$

3. Basis of a subspace (the eigenspaces)

basis for kernels

$$W + I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}, \underline{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

$\lambda = 2$ $\underline{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$ in $\ker(W - 2I)$

G.S.

O.N. for $\lambda = -1$, $\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$, $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$\lambda = 2$ $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$O = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$O^T W O = \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$