$\boxed{1.}$

Eigenvalues

algebraic mult. — geometric mult.

Eigenvalue: A, $n \times n$

real number / scalar λ

$\vec{v}$ ← eigenvector associated to λ

s.t. there exists $\vec{v} \neq \vec{0}$

s.t. $A\vec{v} = \lambda \vec{v}$

$A\vec{0} = \lambda \vec{0}$

cols of $A - \lambda I$ lin. dep.

$\Updownarrow$

$\Leftrightarrow$ $A - \lambda I$ non being invertible

How to find?

λ eigenvalue of A iff $\ker(A - \lambda I) \neq \{\vec{0}\}$

exists $\vec{v} \neq \vec{0}$ in $\ker(A - \lambda I)$

$(A - \lambda I)\vec{v} = \vec{0}$, $A\vec{v} - \lambda\vec{v} = \vec{0}$

$A\vec{v} = \lambda\vec{v}$

λ eval of A

$\Updownarrow$

$A - \lambda I$ not invertible

$\Updownarrow$

$\det(A - \lambda I) = 0$

$\swarrow \quad \downarrow$

characteristic polynomial $\quad$ polynomial in λ

$$\underbrace{\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & & \vdots \\ \vdots & & \ddots & \\ 1 & \cdots & 1 & 2 \end{pmatrix}}_{B} - I = \begin{pmatrix} 1 & \cdots & & 1 \\ 1 & \cdots & & 1 \\ 1 & \cdots & & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & & 1 \end{pmatrix}$$

null $n-1$

rk $= 1$

$$-(n+1)I = \begin{pmatrix} -(n-1) & 1 & \cdots & 1 \\ 1 & -(n-1) & \cdots & 1 \\ \vdots & & & 1 \\ & & \cdots & -(n-1) \end{pmatrix}$$

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ in $\ker(B - (n+1)I)$

$\deg \det(A - \lambda I) = n$

A $n \times n$ matrix

$\lambda^2 - (\operatorname{tr} A)\lambda + \det(A)$

2×2

$\operatorname{tr}(A)$ = sum of eigenvalues
w/ multiplicity

$\det(A)$ = prod of eigenvalues
w/ mult

$(r_1 - \lambda)(r_2 - \lambda) = \det(A - \lambda I)$

r_1, r_2 the eigenvalues of A

$$J_{\underline{n}}(k) = \begin{pmatrix} k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & k \end{pmatrix}$$

n×n

$$\begin{pmatrix} k & 1 \\ & k \end{pmatrix} = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$$

$$\det(\underbrace{J_n(k) - \lambda I}_{J_n(k-\lambda)})$$

$$\det\begin{pmatrix} k-\lambda & 1 & & 0 \\ & k-\lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & k-\lambda \end{pmatrix} = \underbrace{(k-\lambda)^n}$$

$\lambda = k$ <u>only</u> eigenvalue

$$\det \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{44} \end{pmatrix} = a_{11}a_{22}\cdots a_{44}$$

upper Δ

$$\det \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ * & & a_{44} \end{pmatrix} = a_{11}a_{22}\cdots a_{44}$$

$$\sum_{P \text{ patterns}} (\operatorname{sign} P)(\operatorname{prod} P)$$

$$P = (1,1),(2,2),\ldots,(4,4)$$

algebraic multiplicity of an eigenvalue

$$\det(A-\lambda I) = (2-\lambda)^{2}(3-\lambda)(12.5-\lambda)^{100}$$

eigenvalue	alg mult
2	2
3	1
12.5	100

$$\det(J_n(k)-\lambda I) = (k-\lambda)^n$$

only eigenvalue is $\lambda = k$.

alg. mult. of k is n.

geometric mult $\leq$ alg. multiplicity

eigenvector of eigenvalue $\vec{0}$

$$A\vec{v} = 0\vec{v}$$
$$= \vec{0}$$

$$\begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

Something nonzero in kernel

$$rk(A) = n - \underbrace{null(A)}$$

$null(A) \leq$ alg mult of e.v. 0 in char poly

$$\lambda^d (\ldots -)$$

$$\det(A - \lambda I)$$

$$\lambda = 0$$

$$\det(A)$$

$\det(A-\lambda I)$ degree $\underline{n}$ polynomial

$\|$

$\underline{(\underline{r_1}-\lambda)^{e_1} \cdots (\underline{r_m}-\lambda)^{e_m}}$

$e_1+e_2+\dots+e_m=n$

$\text{alg mult}(r_1)+\dots \text{alg m}(e_m)=n$

"sum of algebraic multiplicities is n"

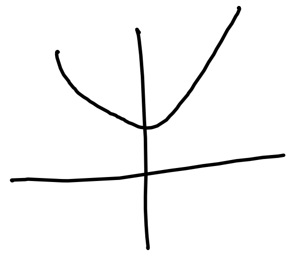
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$(\lambda-i)(\lambda+i)$

$\|$

$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2+1$

no <u>real</u> roots



geometric multiplicity of an eigenvalue of A

$(A - \lambda I)\vec{v} = 0$

$\Updownarrow$

$A\vec{v} = \lambda\vec{v}$

Set of all solution to $A\vec{v} = \lambda\vec{v}$

is $\ker(A - \lambda I)$. " λ-Eigenspace"

geom mult of e.v. λ is $\dim(\ker(A - \lambda I))$

$= \text{null}(A - \lambda I)$

$\lambda = 0$, $\text{null}(A) =$ geom mult of 0.

$J_n(k)$ eigenvalues are $\lambda = k$

geom mult of $\lambda = k$

$$\dim \ker \left(J_n(k) - kI \right)$$

$$\underbrace{\dim \ker}_{\text{null}} \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & 0 & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & 0 & \vdots & & 0 \end{pmatrix}$$

0 lin indep

null $= 1$ $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is ker $\left(J_n(k) - kI \right)$

rk $= n-1$

$J_n(k)$

eigenvalues $\quad k$

alg mult $\quad n$

geom mult $\quad 1$

$1 \le \underline{\text{geom}} \le \underline{\text{alg}} \le n$

λ geo. mult d,

$\vec{v}_1, \ldots, \vec{v}_d$

λ eev.

A similar $\left(\begin{array}{c|c} \lambda \ddots \lambda & * \\ \hline 0 & * \end{array}\right)$

$(\lambda - x)^d$ ~~div~~ factor of char poly

$$\text{geom} \overset{?}{=} \text{alg}$$

it always true → diagonalizable

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eigenvalues 3x3

1, 2, 3

alg	1	1	1
geom	1	1	1

$$\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

ecu 1
am 2
gm 2

diagonalizable

"different eigenvalues => diagonalizable."

n-many

sum of geom mult = 4

geom mult = alg mult for all eigenvalues

⇕

diagonalizable

"Jordan canonical form"

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$(1-\lambda)^2 =$

g.m. 1

a.m. 2

g.m. $\text{null}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$= \text{null}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 2$

$\ker \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2$

例1

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$$

$(1-\lambda)^2(2-\lambda)$

e.v. $\quad 1, 2$

a.m $\quad 2, 1$

g.m $\quad 2, 1$

$(A-\lambda I)$

$\lambda = 1 \quad \begin{pmatrix} 0 & & \\ & 0 & \\ & & 2 \end{pmatrix}$

$\lambda = 2 \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$

$$\text{g.m} = \text{null}(A-\lambda I)$$
$$= n - \text{rk}(A-\lambda I)$$

DA $\quad A, \quad B$

same e.v.

same a.m $\implies A \underset{\text{sim}}{\sim} B$

and g.m = a.m.

$$A \sim D$$

$$B \sim D$$

$$S^{-1} A S = D$$

$$T^{-1} B T = D$$

$$\begin{aligned} B &= T D T^{-1} \\ &= T S^{-1} A S T^{-1} \end{aligned}$$

2. only e.v. = 0

a.m = 4

char λ^4

	e.v.	g.m	a.m
A	0	2	4
B	0	2	4

$$A^2 = 0$$

$$B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix} \neq 0$$

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix}$$

$Ae_1 = 0$

$\cancel{A} e_2 = e_1$

$Ae_k = e_{k-1}$

$$J_3(0) \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \leftarrow J_1(0)$$

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)^2 = \left(\begin{array}{c|c} A^2 & 0 \\ \hline 0 & B^2 \end{array}\right)$$

$$\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&\end{pmatrix}^2 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}&1\\&&\end{pmatrix}$$

$$\begin{pmatrix}e_1 \mapsto e_3\\ e_4 \mapsto 0\\ e_3 \mapsto e_2\end{pmatrix} \quad e\ (e_1, e_3, e_2)$$

$Av = \lambda v$

$v \neq 0$

$A^{100} v = \lambda^{100} v$

$Av = \lambda v \quad \lambda \neq 0 \quad v \neq 0$

$A^n v = \lambda^n v \neq 0$

$$\begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} & 1 \\ & \\ & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\begin{array}{l} v_1 \quad v_2 \quad v_3 \\ Av_2 = v_1, \quad Av_3 = 0 \\ Av_1 = A^2 v_2 = 0 \end{array}}$

$$\begin{pmatrix} & 1 \\ & \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$$

1

2

Cayley-Hamilton theorem $\to$ $p(\lambda) = \det(A - \lambda I)$

$p(A) = 0$

$\updownarrow$

minimal polynomial of a matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \lambda^2 + 1$$

$R^2 + I$

$R^2 = -I$

$R^2 + I = 0$

3×3

$$A^4 = 0$$

$$\Downarrow$$

$$A^3 = 0$$

$p(\lambda) = \det(A - \lambda I)$

it n factors

$$A \sim \begin{pmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_l}(\lambda_l) \end{pmatrix}$$

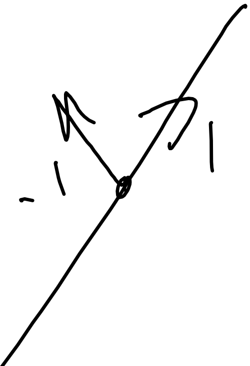
"unique"

up to reordering the blocks

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^{d} \qquad \begin{pmatrix} \overbrace{c \quad c \cdots c}^{d} & 1 & \\ \ddots & \ddots & 1 \\ & & \end{pmatrix}$$

$$\begin{pmatrix} c & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \det = -1$$

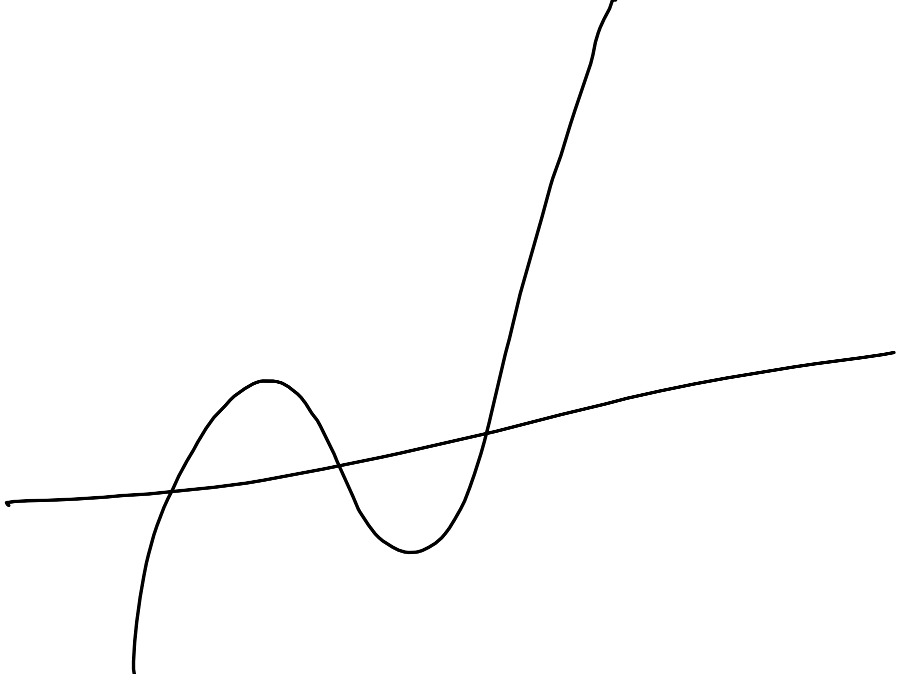


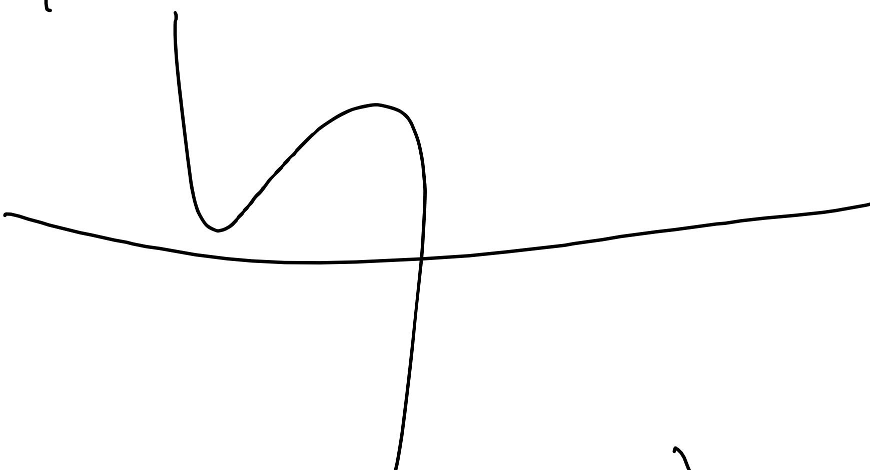
$$\lambda^2 - 1$$

$$(\lambda - 1)(\lambda + 1)$$

$$\begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$





$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$(\lambda^2+1)^n$

$a+bi, \quad a-bi$

A, B

$S^{-1}AS = B$

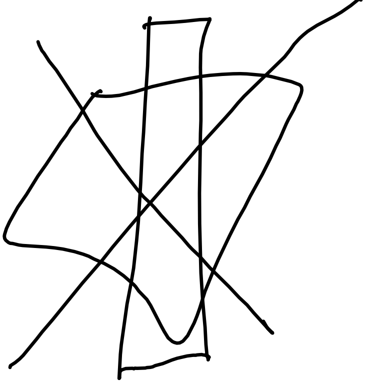
$B^n = (S^{-1}AS)^n = S^{-1}AS\,S^{-1}AS\,S^{-1}AS \cdots S^{-1}AS$

$= S^{-1}A^nS$

$p \qquad p(S^{-1}AS) = S^{-1}p(A)S$

A, B similar $\Rightarrow$ A^n, B^n simi

$p(A), p(B)$

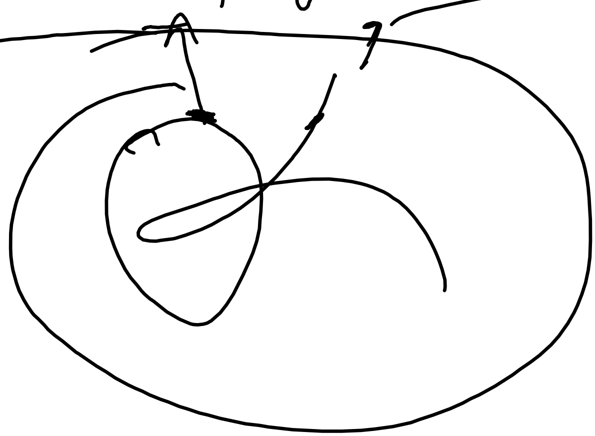


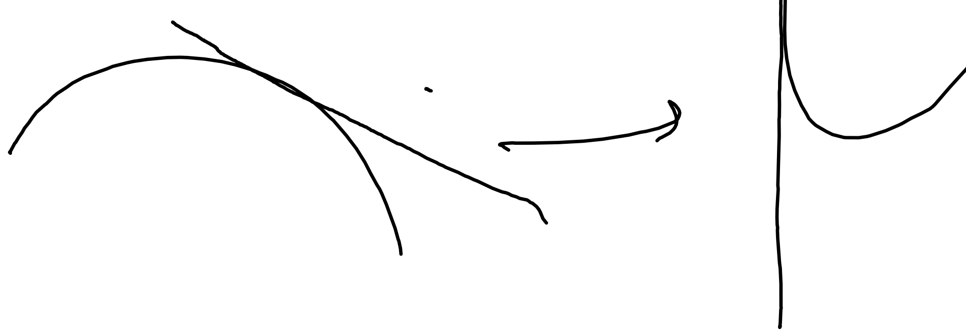
A $n \times n$

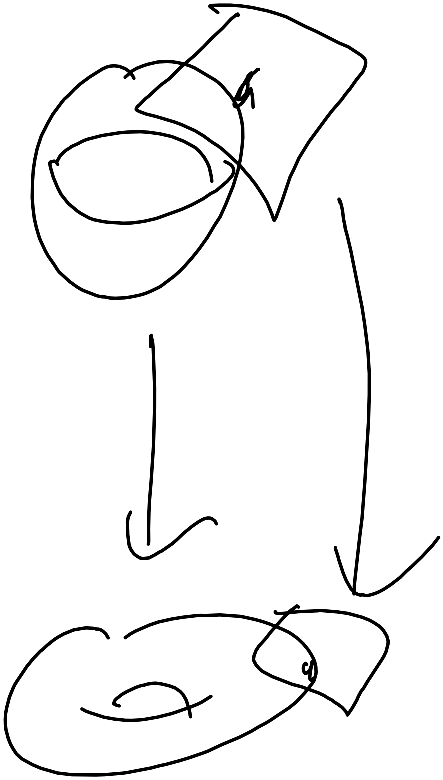
invertible

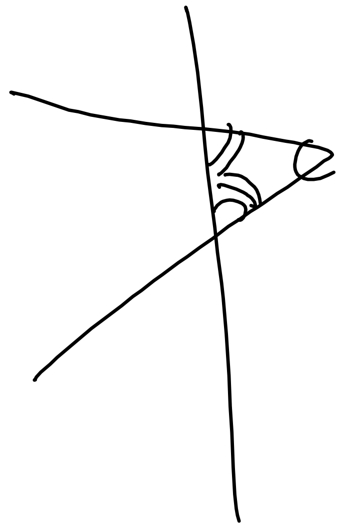
$\left.\begin{array}{l} \text{null } A = 0 \\ \text{or} \\ \text{rk} = n \end{array}\right\}$

$$(x^2+y^2-1)(y^2-x^2(x-1))=0$$









$x^n + y^n = z^n \qquad n \geq 3$

Andrew Wiles

$x^n + y^n = z^n \qquad n \geq 3$

has no positive integer sols (x, y, z)

$n = 2 \qquad \left(\frac{x}{z}\right)^n + \left(\frac{y}{z}\right)^n = 1$

$(3, 4, 5) \qquad n = 2$

$(5, 12, 13)$