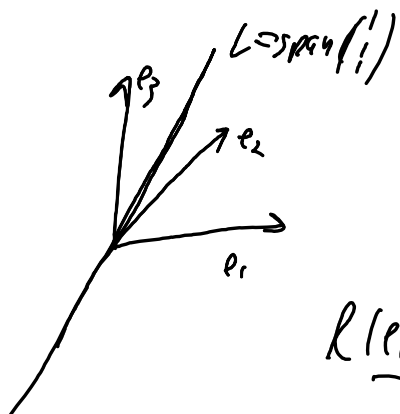
Eigen-things

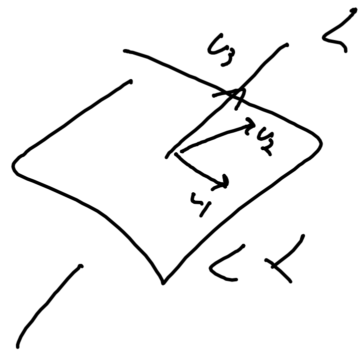
German

eigen - innate, of itself

Recall change of basis

rotate about $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in $\mathbb{R}^3$





$R(\vec{e_1}) = ?$

$R(\vec{e_2}) = ?$

$R(\vec{e_3}) = ?$

$\begin{pmatrix} ? \\ \sqrt{2} \\ ? \end{pmatrix}$ $\quad$ ([illegible])

in this basis $\vec{L}$ → v_1, v_2, v_3

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This transformation

deserves better than standard basis

$\underbrace{v_1, v_2}_{\text{perp to } v_3}, \underbrace{v_3}_{\text{eigenvector}}$ naturally suited to the problem

$$A = \begin{pmatrix} 1/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix}$$

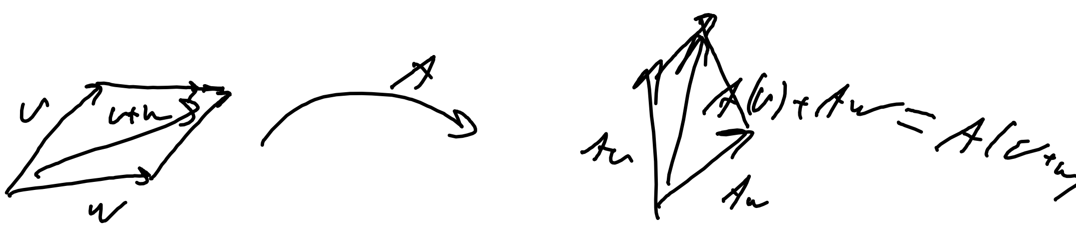
$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

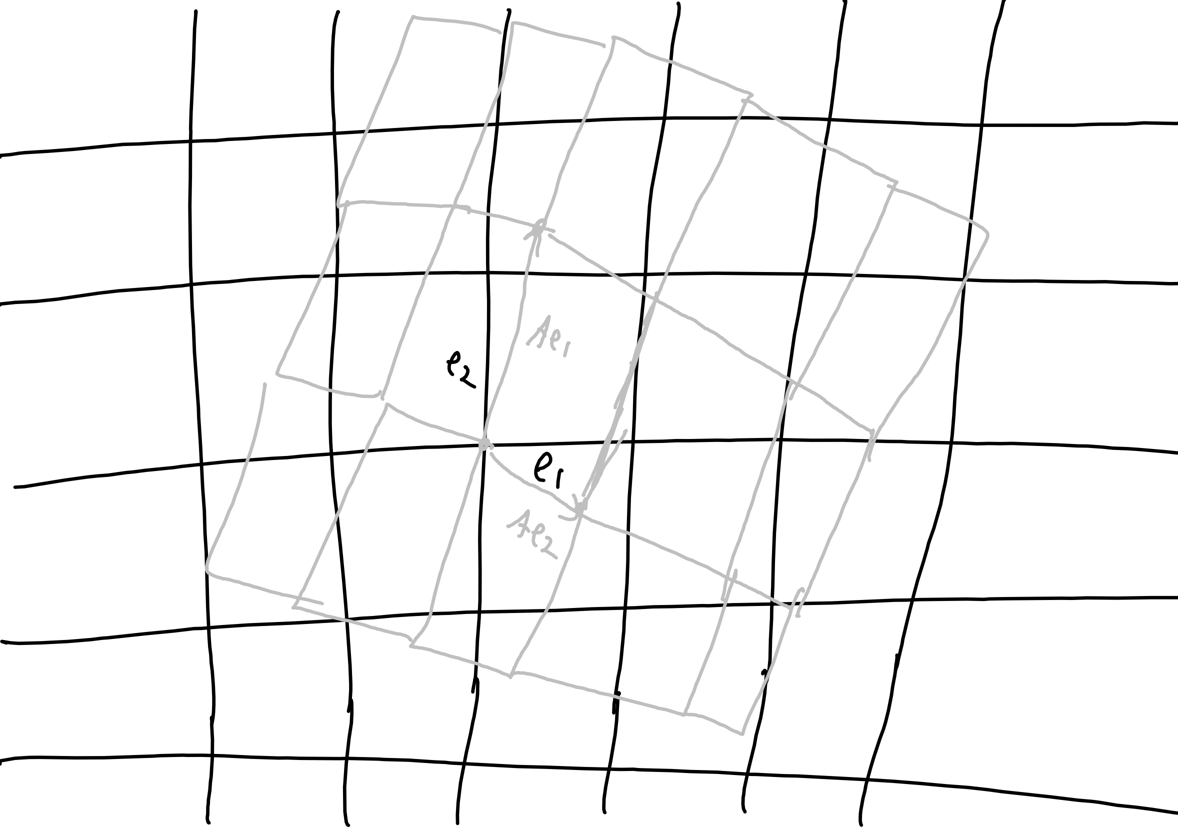
$$A\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -1\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\vec{v} \longmapsto A\vec{v}$$

linear $\longleftrightarrow$ send evenly spaced grid to an evenly spaced grid





gray grid — what the standard grid is transformed to under A

black grid — standard grid
e_1, e_2

disjointed from itself
dissonant

$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}$

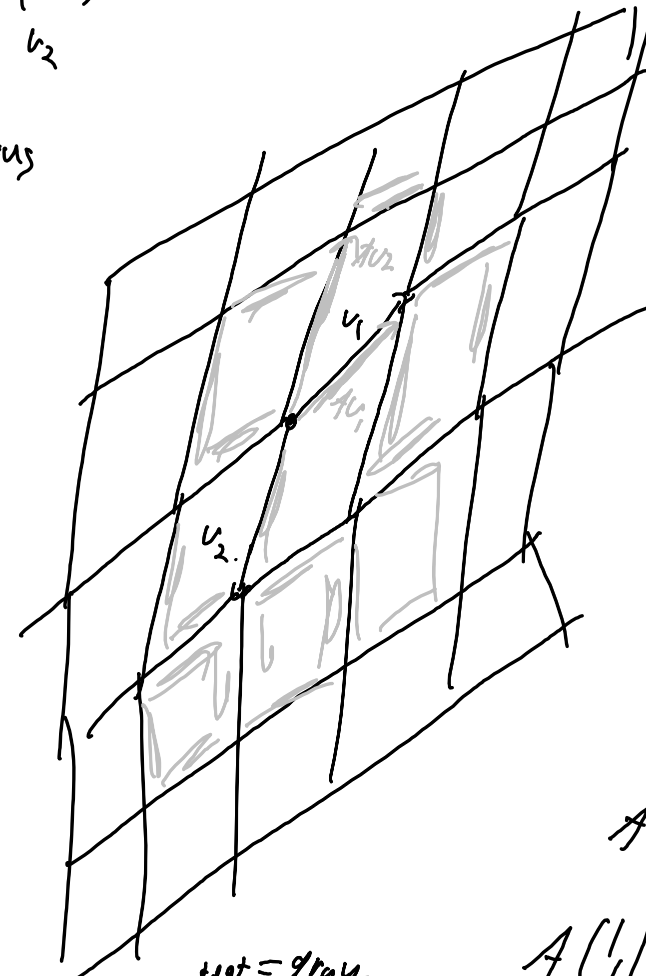
$v_1 \quad v_2$

harmonious

$\perp$

geom. nicety

output grid is aligned w/ input grid



input = black, output = gray

black grid — $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}$

$Av_1 = v_1$

$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

$A\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$

$Av_2 = -v_2$

e.v. $= -1$

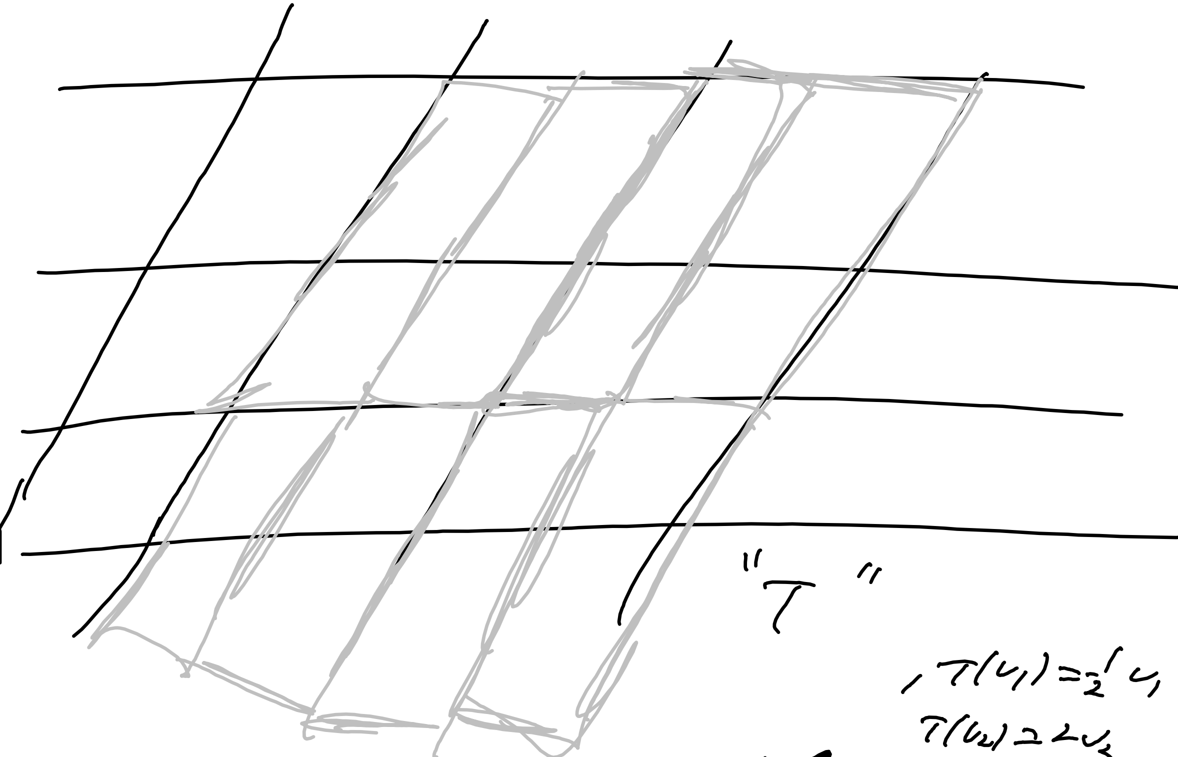
reflects v_2

fixes v_1

e.v. $= 1$

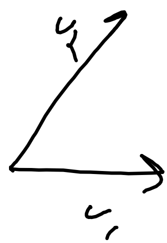
in the basis v_1, v_2, A repr as $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

alg. nicety, ~~better~~ simpler matrix

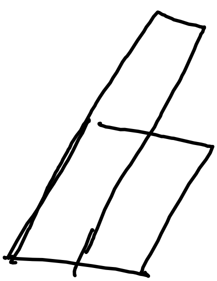


"T"

$T(v_1) = \frac{1}{2} v_1$
$T(v_2) = 2 v_2$



harmonic



eiv. v_1 $\frac{1}{2}$
eiv. $v_2 = 2$

$\boxed{\text{diagonalizing}}$

algebra easier – diagonal matrices nice

geometry easier – lined up correctly

$$A = \begin{pmatrix} 1/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix}$$

how did I determine $A\begin{pmatrix}1\\1\end{pmatrix} = 1\begin{pmatrix}1\\1\end{pmatrix}$

$A\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix} = -\begin{pmatrix}-1\\2\end{pmatrix}$

how did I find those vectors?

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find eigenvalues

$\det(A - \lambda I)$. polynomial in λ

roots of this are precisely the eigenvalues

$$\det\begin{pmatrix} 1/3 - \lambda & 2/3 \\ 4/3 & -1/3 - \lambda \end{pmatrix}$$

$$= \left(\frac{1}{3} - \lambda\right)\left(-\frac{1}{3} - \lambda\right) - \frac{2}{3} \cdot \frac{4}{3}$$

$$= \frac{-1}{9} + \lambda^2 - \frac{8}{9}$$

$$= \lambda^2 - 1$$

$$= (\lambda - 1)(\lambda + 1)$$ → eigenvalues are -1, 1.

$\lambda=-1$ $\quad \det(A+I)=0$

$A+I$ has a nonzero kernel

$\ker(A+I) \neq \{\vec{0}\}$

bigger

$\ker(A+I)$

$$\begin{pmatrix} 4/3 & 2/3 \\ 4/3 & 2/3 \end{pmatrix}$$

row reduce

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$\ker(\cdot) = \underbrace{\operatorname{span} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$

$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ eigenvector of eigenvalue -1

$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in $\ker(A+I)$

$(A+I)\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \vec{0}$ $\qquad A\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$A\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} =$

$\lambda = 1 \quad \ker(A - I)$

$$\ker \begin{pmatrix} -2/3 & 2/3 \\ 4/3 & -4/3 \end{pmatrix} = \ker \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

row reduction $\parallel$

$$\operatorname{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Concl. $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector for A of eigenvalue 1

$\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ eigenvector for A of eigenvalue -1

$\mathcal{B} = (\vec{v}_1, \vec{v}_2)$

$$S = \begin{array}{c} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \\ v_1 \quad v_2 \end{array} \quad \left(= {}_{\mathcal{E}}[I]_{\mathcal{B}} \right)$$

$\mathcal{E} = (e_1, e_2)$

$$S^{-1} A S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

diagonalized

Procedure

almost always how you find eigenvectors and eigenvalues, and hence how you diagonalize

Setup. A $n \times n$ matrix

1. Compute $\det(A - \lambda I)$, "characteristic polynomial" poly of deg n in λ

2. find the roots / zeroes of $\det(A - \lambda I)$

$\updownarrow$

find eigenvalues of A

3. for each eigenvalue λ, compute $\underbrace{\ker(A - \lambda I)}$

using, say, RREF.

λ-eigenspace
solutions $\vec{v}$ to $A\vec{v} = \lambda\vec{v}$

4. Find a basis for $\ker(A - \lambda I)$

$\updownarrow$ (RREF)

eigenvectors

5. Assemble those bases, form the change of basis matrix, and diagonalize $S^{-1}AS = D$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

rows: lin dep

$\ker(A)$ 2d

1. $\det(A - \lambda I)$

$$\begin{array}{ccccc} 1-\lambda & 2 & 3 & 1-\lambda & 2 \\ 2 & 4-\lambda & 6 & 2 & 4-\lambda \\ 3 & 6 & 9-\lambda & 3 & 6 \end{array}$$

$$(1-\lambda)(4-\lambda)(9-\lambda) + 2\cdot 6\cdot 3 + 3\cdot 2\cdot 6$$
$$- \left(2\cdot 2\cdot (9-\lambda) + (1-\lambda)\cdot 6\cdot 6 + 3(4-\lambda)\cdot 3 \right)$$

$$= \underline{\lambda}^2 (14 - \lambda)$$
(exc.)

2. roots? $\lambda = 0$, $\lambda = 14$

3. $\lambda = 0$. $\ker(A - 0I) = \ker(A) = \ker \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\lambda = 14$

$\ker(A - 14I) = \begin{pmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{pmatrix}$

(ecrr) ||

Span $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in $\ker(A - 14I)$

$\begin{array}{c} v_1 \\ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{array}$ eigen vect eigenvalu 14

$\begin{array}{cc} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, & \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \\ v_2 & v_3 \end{array}$ eigenvech eigenvalu 0

$B = (v_1, v_2, v_3)$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$[A]_B = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Projection times 14

$A =$ lin. orthog proj onto line spanned by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Geometric examples

Reflection about line L in $\mathbb{R}^2$

$L = \text{span}(\vec{v})$ $\quad \vec{v} \neq \vec{0}$

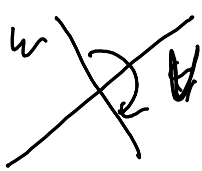
$\vec{w}$ perp $\vec{v}$ $\quad , \vec{w} \neq \vec{0}$

$\vec{v}$ eigenvect e.v. 1

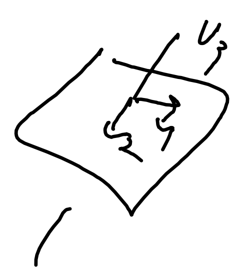
$\vec{w}$ eigenvect of e.v. -1

in this "eigenbasis", $L \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\quad v \quad w$



Refl about plane in $\mathbb{R}^3$



v_1, v_2 $\qquad v_3 = v_1 \times v_2$

e.v. 1 $\quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ $\quad$ e.v. -1

orthog proj onto V

eiv. 1 — $v_1, \ldots, v_k$ basis for V

eiv. 0 — $v_{k+1}, \ldots, v_n$ basis for $V^\perp$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

$\underbrace{v_1 \ldots v_k}$ $\underbrace{v_{k+1} \ldots v_n}$

shears
reflections } not diagonalizable

$\boxed{\text{Aside}}$

$$\det \begin{pmatrix} 2 & 1 & \cdots & & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ & & & 2 & 1 \\ 1 & \cdots & & 1 & 2 \end{pmatrix}$$

2 diag
1 elsewhere

$\underline{\ker(A - I)}$ $= \ker \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{pmatrix}$

rank $\underline{1}$, im $=$ span $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$

$\underline{\text{null} = n-1}$

$\det A =$ product of eigenvalues
" w/ multiplicity
algebraic

$p(\lambda) = \det(A - \lambda I)$

$p(0) = \det(A)$
||
product of roots of p $=$ prod. of eigenvalues

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

4×4

$$A - (n+1)I$$

$$\begin{pmatrix} -(n-1) & 1 & \cdots & 1 \\ 1 & -(n-1) & 1 \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & & (-n-1) \end{pmatrix}$$

sum along any row is 0

$$(A - (n+1)I)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \vec{0}$$

eigenvector of eigenvalue $n+1$

$$\det(A) = n+1$$