Determinants

Def. $\displaystyle\sum_{\text{patterns } P} (\operatorname{sgn} P)(\operatorname{prod} P)$

$\left.\begin{array}{l} 2\times 2 \\ 3\times 3 \end{array}\right\}$ often just apply this formula

upper (lower) triangular

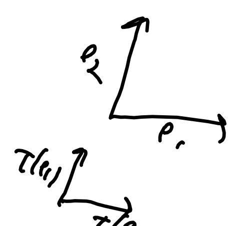
$\det\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} = 1\cdot 2\cdot 3 = 6$

$\det\begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{pmatrix} = 1\cdot 2\cdot 3 = 6$

Gauss-Jordan

$A \xrightarrow[i\neq j]{R_i \mapsto R_i + \lambda R_j} B \qquad \det(A) = \det(B)$

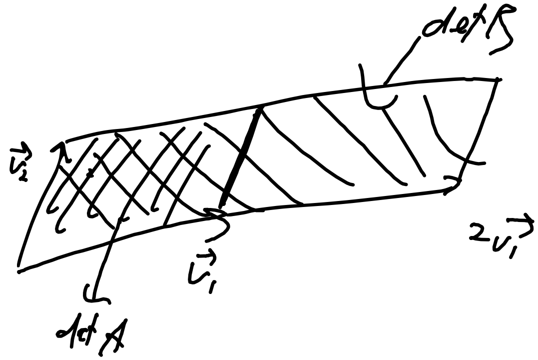
$A \xrightarrow[i\neq j]{R_i \leftrightarrow R_j} B \qquad \det(A) = -\det(B)$



$$A \xrightarrow{R_i \mapsto \lambda R_i} B \qquad \underbrace{\det(B)} = \underbrace{\lambda \det(A)}$$

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} \longmapsto \begin{pmatrix} R_1 \\ 5R_2 \\ R_3 \end{pmatrix}$$

$\det A$ $\qquad$ $\det B = 5 \det \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}$



$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - 4R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\xrightarrow{R_3 - 7R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

rows are lin. dep.

$$\xrightarrow[R_2 \mapsto \left(-\frac{1}{3}\right) R_2]{\left(-\frac{1}{3}\right)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\xrightarrow{R_3 \mapsto R_3 + 6R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

det? $= 0$

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow[\left(\frac{1}{2}\right)]{R_1 \mapsto \frac{1}{2} R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} = B$$

$$\xrightarrow[(-1)]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} = C \xrightarrow[(1)]{R_2 \mapsto R_2 - R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} = D$$

$$\xrightarrow{R_3 \mapsto R_3 + R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix} = E$$

$$\det(\underbrace{\text{last matrix}}_{E}) = -4$$

$\det(B) = \frac{1}{2}\det(A)$

$\det(C) = -\det(B)$

$\det(D) = \det(C)$

$\det(E) = \det(D)$

$$\det(E) = -\frac{1}{2}\det(A)$$

$$\parallel$$

$$\det(A) = 8$$

Expansion by Minors

$$A \, \underline{\mathrm{adj}(A)} = \det(A) \, \underline{I_n}$$

$$\left(A \, \mathrm{adj}(A)\right)_{\underline{ii}} = \sum_{k=1}^{n} A_{ik} \, \mathrm{adj}(A)_{ki}$$

$$= \sum_{k=1}^{n} A_{ik} \, (-1)^{k+i} \det(A_{ik})$$

$= \underline{\text{expansion}}$ about $\underline{i^{th} \text{ row}}$ for $\det A$

$$\mathrm{adj}(A) \, A = \det(A) \, I_n$$

$\Downarrow$

$\underline{\text{expansion}}$ about $\underline{\text{columns}}$

expanding about row i "minor" of A

- $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \boxed{\det(A_{ij})}$

expand about jth col

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

A_{ij} = "submatrix" given by deleting row i and col j

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$

$$\det(A_{11}) = 45 - 48 = -3$$

Recursion

$n \times n$ det, A

$(n-1)\times(n-1)$ det A_{ij}, n many

$$\begin{pmatrix} + & - & + & - & + & - \\ - & + & - & & \ddots & \\ + & - & & \ddots & & \\ - & & & & & \\ + & & \ddots & & \ddots & \\ - & & & & & \end{pmatrix}$$

$$\det\begin{pmatrix} \underline{0} & 7 & 5 & 3 \\ \underline{1} & 1 & 2 & 1 \\ \underline{1} & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

expand about 1^{st} column

$$= \cancel{0 \cdot \det\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix}} - 1 \det\begin{pmatrix} 7 & 5 & 3 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix} \qquad \underline{(i)\ 17}$$

$$0 \qquad + 1 \det\begin{pmatrix} 7 & 5 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad \underline{(ii)\ 13}$$

$$- 1 \det\begin{pmatrix} 7 & 5 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix} \qquad \underline{(iii)\ -2}$$

$$-1(17) + 1(13) - 1(2)$$

$$= -17 + 13 - 2 = -17 + 11 = -7 + 1 = -6$$

i) $\det\begin{pmatrix} 7 & 5 & 3 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix}$

expand about 2^{nd} row

$$-1\cdot\det\begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix} = -1(10-3) = -7$$

$$+2\cdot\det\begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix} = 2\cdot(14-3) = 22$$

$$-(-1)\cdot\det\begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix} = 1\cdot(7-5) = 2$$

$$-7+22+2$$

$$= 17$$

ii) $\det\begin{pmatrix} 7 & 5 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & \underline{2} \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

expand about 3rd column

$3 \cdot \det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = 3(1-2) = -3$

$-1 \cdot \det\begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix} = -1(7-5) = -2$

$+2\det\begin{pmatrix} 7 & 5 \\ 1 & 2 \end{pmatrix} = 2\cdot(14-5) = 18$

$-3-2+18 = -5+18 = 13$

iii) $\det\begin{pmatrix} 7 & 5 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix}$ $\quad\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

from 3rd row

$1 \cdot \det\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = -1$

$-2 \det\begin{vmatrix} 7 & 3 \\ 1 & 1 \end{vmatrix} = -2(7-3) = 8$

$+(-1) \cdot \det\begin{pmatrix} 7 & 5 \\ 1 & 2 \end{pmatrix} = -1(14-5) = -9$

$-1 + 8 - 9 = -2$

1. $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ a_0^3 & a_1^3 & \cdots & a_n^3 \\ \vdots & & & \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix}$ $(n+1)(n+1)$ matrix

$a_0, \ldots, a_n$ some numbers

a) $\underline{n=1.}$ $\begin{pmatrix} 1 & 1 \\ a_0 & a_1 \end{pmatrix}$ $\det = \underline{a_1 - a_0}$

$\underline{n=2.}$ $\begin{pmatrix} 1 & 1 & 1 \\ a_0 & a_1 & a_2 \\ a_0^2 & a_1^2 & a_2^2 \end{pmatrix}$

expand about 3rd col.

$$1 \underline{\det \begin{pmatrix} a_0 & a_1 \\ a_0^2 & a_1^2 \end{pmatrix}} - a_2 \underline{\det \begin{pmatrix} 1 & 1 \\ a_0^2 & a_1^2 \end{pmatrix}}$$

$$+ a_2^2 \underline{\det \begin{pmatrix} 1 & 1 \\ a_0 & a_1 \end{pmatrix}}$$

$$(a_0 a_1^2 - a_1 a_0^2) - a_2 (a_1^2 - a_0^2) + a_2^2 (a_1 - a_0)$$

$$\underline{= (a_2 - a_0)(a_2 - a_1)(a_1 - a_0)}$$

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_n \\ \vdots & & \vdots \\ a_0^n & \cdots & a_n^n \end{pmatrix} = \prod_{i<j} (a_j - a_i)$$

$\boxed{\text{claim:}}$

$(a_1 - a_0)(a_2 - a_0) \cdots (a_n - a_0)$.

$(a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1)$

$\vdots$

$(a_n - a_{n-1})$

$a_n = t$

$$b)\ \det\begin{pmatrix} 1 & \cdots & 1 & 1 \\ a_0 & \cdots & a_{n-1} & t \\ \vdots & \ddots & & \vdots \\ a_0^n & \cdots & a_{n-1}^n & t^n \end{pmatrix} = f(t)$$

expand about $n+1^{st}$ column

$$(-1)^{1+(n+1)} 1 \boxed{\det\begin{pmatrix} a_0 & \cdots & a_{n-1} \\ \vdots & \ddots & \vdots \\ a_0^n & \cdots & a_{n-1}^n \end{pmatrix}}^{\text{last}} + (-1)^{n+3}\, t \det\begin{pmatrix} 1 & \cdots & 1 \\ a_0^2 & \cdots & a_{n-1}^2 \\ \vdots & \ddots & \vdots \\ a_0^n & \cdots & a_{n-1}^n \end{pmatrix}$$

$(-1)^{n+2}$

$+(-1)^{n+4} t^2 \det(\ \ ) \cdots$

$\downarrow$

a_i's

$$+(-1)^{(n+1)+(n+1)}\, t^n \det\begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_{n-1} \\ \vdots & & \vdots \\ a_0^{n-1} & \cdots & a_{n-1}^{n-1} \end{pmatrix}$$

recursively understand

(mathematical induction)

$f(t)$

Mark: a polynomial in t

$\deg = n$

Note: $f(a_0) = 0$

$$f(a_0) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_0 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_0^n \end{pmatrix} = 0$$

lin alg $\Rightarrow$ $f(a_1) = 0$

$\vdots$

$f(a_{n-1}) = 0$

$$f(t) = k \prod_{i=0}^{n-1} (t - a_i), \quad f(a_n) = \det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_n \\ \vdots & & \vdots \\ a_0^n & \cdots & a_n^n \end{pmatrix}$$

algebra //

what is k?

$$f(0) = k \prod_{i=0}^{n-1} (-a_i)$$

$$= (-1)^n k \prod_{i=0}^{n-1} a_i$$

$p(t) = b_0 + b_1 t + \dots + b_n t^n$

$p(0) = b_0$ constant coefficient

$$\underline{f(0)} = (-1)^{n+2} \det \begin{pmatrix} a_0 & \cdots & a_{n-1} \\ \vdots & & \vdots \\ a_0^n & \cdots & a_{n-1}^n \end{pmatrix}$$

$$= (-1)^{n+2} a_0 \det \begin{pmatrix} 1 & a_1 & \cdots & a_{n-1} \\ a_0 & \vdots & & \vdots \\ \vdots & & & \\ a_0^{n-1} & a_1^n & \cdots & a_{n-1}^n \end{pmatrix}$$

$\vdots$

$$= \underline{(-1)^{n+2} a_0 \cdots a_{n-1}} \det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_{n-1} \\ \vdots & & \vdots \\ a_0^{n-1} & & a_{n-1}^{n-1} \end{pmatrix}$$

compare: $\underline{(-1)^n} k \underline{\prod_{i=0}^{n-1}} a_i$

$$(-1)^{n+2} = (-1)^n (-1)^2 = (-1)^n$$

$$\underline{(-1)^{n+2}} \; \underline{a_0 \cdots a_{n-1}} \det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_{n-1} \\ \vdots & & \\ a_0^{n-1} & \cdots & a_{n-1}^{n+1} \end{pmatrix} = \underline{(-1)^n} \underline{k} \underline{\prod_{i=0}^{n-1}} a_i$$

$$k = \det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_{n-1} \\ \vdots & & \\ a_0^{n-1} & \cdots & a_{n-1}^{n-1} \end{pmatrix}$$

$a_i = a_j \quad i \neq j$

$$\det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_n \\ \vdots & & \\ a_0^n & \cdots & a_n^n \end{pmatrix} = 0$$

repeated colums

assume $a_0, \ldots, a_n$ all distinct

no repetition

at most one $a_i = 0$.

if $a_i = 0$, swap w/ a_n

$a_1 = 0$. swap 1^{st} $n+1^{st}$ col

assume that $a_i \neq 0$

$$k = \det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_{n-1} \\ \vdots & & \\ a_0^{n-1} & \cdots & a_{n-1}^{n-1} \end{pmatrix}$$

$$= \prod_{1 \leq i < j \leq n-1} (a_j - a_i)$$

$$f(t) = \cancel{k} \prod_{i=0}^{n-1} (t - a_i)$$

$$= \prod_{i<j\le n-1} (a_j - a_i) \prod_{k=0}^{n-1} (t - a_k)$$

$$\underline{f(a_n)} = \underbrace{\prod_{i<j\le n-1} (a_j - a_i)}_{\underline{k}} \,\big/\, (a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})$$

$$\| \qquad\qquad \|$$

$$\det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_n \\ \vdots & & \\ a_0^n & \cdots & a_n^n \end{pmatrix} = \prod_{1\le i<j\le n} (a_j - a_i)$$

$n=5$

$$\det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_n \\ \vdots & & \vdots \\ a_0^n & \cdots & a_n^n \end{pmatrix} \overset{?}{=} \underbrace{\det \begin{pmatrix} 1 & \cdots & 1 \\ a_0 & \cdots & a_{n-1} \\ \vdots & & \\ a_0^{n} & \cdots & a_{n-1}^{n-1} \end{pmatrix}}_{!\,?\,!} (a_n - a_0)(a_n - a_1)\cdots(a_n - a_{n-1})$$

$$n=3, \quad \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_0 & a_1 & a_2 & a_3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 \end{pmatrix} = \underbrace{\det \begin{pmatrix} 1 & 1 & 1 \\ a_0 & a_1 & a_2 \\ a_0^2 & a_1^2 & a_2 \end{pmatrix}}\, (a_3 - a_0)(a_3 - a_1)(a_3 - a_2)$$

$$\underbrace{\det \begin{pmatrix} 1 & 1 \\ a_0 & a_1 \end{pmatrix}}_{(a_1 - a_0)} (a_2 - a_1)(a_2 - a_0) \cdot$$

$n=3.$ $\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ a_0 & a_1 & a_2 & a_3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 \end{pmatrix}$ $= \det\begin{pmatrix} 1 & 1 & 1 \\ a_0 & a_1 & a_2 \\ a_0^2 & a_1^2 & a_2^2 \end{pmatrix}(a_3-a_0)(a_3-a_1)(a_3-a_2)$

$\det\begin{pmatrix} 1 & 1 \\ a_0 & a_1 \end{pmatrix}(a_2-a_1)(a_2-a_0).$

a_1-a_0

$$\det\begin{pmatrix} 1 & 1 & 1 \\ a_0 & a_{\cancel{2}} & a_3 \\ a_0^2 & a_1^2 & a_2^2 \end{pmatrix}$$

$$= (a_1-a_0)\cdot(a_2-a_1)(a_2-a_0)$$

$$= \Big((a_1-a_0)(a_2-a_1)(a_2-a_0)\Big)(a_3-a_0)(a_3-a_1)(a_3-a_2)$$

$$= \prod_{1\le i<j\le 3}(a_j-a_i)$$