Determinants

A $n \times n$ matrix

Def. $\displaystyle\sum_{\substack{\text{patterns}\\ P}} \underbrace{(\operatorname{sgn} P)}_{(-1)^{\#\text{inversion}}} \underbrace{(\operatorname{prod} P)}_{\prod_{i=1}^{n} a_{i\,p(i)}} = \det(A)$

$$\begin{array}{c} 11 \\ 23 \\ 32 \end{array} \begin{pmatrix} 0 & & \\ & & 0 \\ & 0 & \end{pmatrix}$$

$P \quad (1,1),\ \underbrace{(2,3),\ (3,2)}$

$\operatorname{sgn}(P) = -1$

$\operatorname{prod}(P) = a_{11}\, a_{23}\, a_{32}$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$

$$\begin{array}{ccccc} + & + & + & & \\ a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ - & - & - & & \end{array}$$

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & & \end{pmatrix}$ $\quad aei + bfg + cdh - bdi - afh - ceg$

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = ??$$

24 terms

24 distinct patterns

$$\sum_{\text{patterns}} \qquad \sum_{\text{24 things}}$$

$\det(5\times5) = ??$ 120 terms

5 mult.

$\det(6\times6) = ??$ 720 terms

6 mult.

$\det(n\times n)$ $n!$ terms

$\boxed{\text{Properties}}$ A, B $n \times n$ matrices

$$\det(AB) = \det(A)\det(B)$$

$$\det(A+B) \neq \det(A) + \det(B)$$

$$\det(A^t) = \det(A)$$

$\det(A) \neq 0 \iff A$ invertible $\quad \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$

Upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots \; 0 & a_{nn} \end{pmatrix}$$

$$\det(A) = a_{11} a_{22} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Need A upper triangular $\quad \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

$$\det\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} = 1 \cdot 2 \cdot 3 = 6$$

$$\det\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = 3$$

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 1 \cdot 2 \cdot 3 = 6$$

$$\boxed{\text{Computation}}$$

Gauss-Jordan elimination

$$A \xrightarrow[\underline{i \neq j}]{R_i \mapsto R_i + \lambda R_j} \underline{B}$$

$$A = \begin{pmatrix} - & R_1 & - \\ - & R_i & - \\ & \vdots & \\ - & R_j & - \\ - & R_n & - \end{pmatrix} \longrightarrow \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & \underline{R_i + \lambda R_j} & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \\ - & R_n & - \end{pmatrix} = B$$

$$\det(B) = ?$$

$$= \det \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_n & - \end{pmatrix} + \det \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & \lambda R_j & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \\ - & R_n & - \end{pmatrix}$$

$$= \det(A) \qquad + \quad 0$$

$$= \det(A)$$

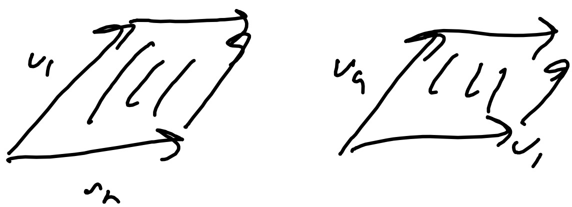
$$A \xrightarrow{R_i \longleftrightarrow R_j} B$$

$$\det(A) = -\det(B)$$

$$\sum_{p \text{ pattern}} (\text{sgn } p)(\text{prod } p)$$

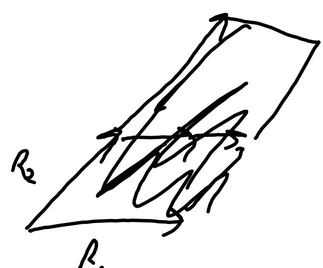
swap $\longrightarrow$ add a swap to each p

$$\text{sgn } p = (-1)^{\# \text{swaps}}$$



Scaling?

$$A \xrightarrow{R_i \mapsto \lambda R_i} B$$



$$\det(B) = \lambda \det(A)$$

$$\det \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & \lambda R_i & - \\ & \vdots & \\ - & R_n & - \end{pmatrix} = \lambda \det \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_n & - \end{pmatrix}$$

det of → ?

$$A = \begin{pmatrix} 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \xrightarrow[(-1)]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix} = B$$

$$\xrightarrow[R_3 \mapsto R_3 - R_1]{(1)} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 0 & -2 \\ 1 & 1 & 1 & 2 \end{pmatrix} = C$$

$$\xrightarrow[R_4 \mapsto R_4 - R_1]{(1)} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{pmatrix} = D \xrightarrow[R_3 \leftrightarrow R_4]{(-1)} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} = E$$

$$A \xrightarrow[-1]{R_1 \leftrightarrow R_2} B \xrightarrow[1]{R_3 \mapsto R_3 - R_1} C \xrightarrow[1]{R_4 \mapsto R_4 - R_1} D \xrightarrow[-1]{R_3 \leftrightarrow R_4} E$$

$$\left.\begin{aligned} \det(B) &= -\det(A) \\ \det(D) &= \det(C) = \det(B) \\ \det(E) &= -\det(D) \end{aligned}\right\} (-1)^2 \det(A) = \det(E)$$

$$\det(A) = \det(E) = 1 \cdot 7 \cdot (-1) \cdot (-2) = 14$$

$$A \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow[\left(\frac{1}{2}\right)]{R_1 \mapsto \frac{1}{2} R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} B$$

$$\xrightarrow[(-1)]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} C \xrightarrow[(1)]{R_2 \mapsto R_2 - R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} D$$

$$\xrightarrow[(1)]{R_3 \mapsto R_3 - R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} E$$

$$A \xrightarrow{\left(\frac{1}{2}\right)} B \xrightarrow{(-1)} C \xrightarrow{(1)} D \xrightarrow{(1)} E$$

$$\left.\begin{aligned} \det(B) &= \tfrac{1}{2}\det(A) \\ \det(C) &= -\det B \\ \det(E) &= \det(D) = \det(C) \end{aligned}\right\} = -\frac{1}{2}\det(A) = \det(E)$$

$$= 1 \cdot 1 \cdot 2$$

$$= 2$$

$$\det(A) = -4$$

$$-\frac{1}{2}\det(A) = \det(E)$$

$$\underbrace{\begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix}}_{P_2} \underbrace{\begin{pmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{pmatrix}} = \begin{pmatrix} - & R_1 & - \\ - & 2R_2 & - \\ - & R_3 & - \end{pmatrix}$$

$\boxed{\text{Expansion by Minors}}$

Recursive method $\quad A \quad n \times n$ matrix

"expanding about column j"

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \underline{a_{ij}} \underbrace{\det(A_{ij})}_{ij \text{ minor}}$$

A_{ij} = omit row i and col j from A

$(n-1) \times (n-1)$ matrix

"expand about row i"

$$\det(A) = \sum_{j=1}^{n} \underline{(-1)^{i+j}} \, \underline{a_{ij}} \det(A_{ij})$$

$$\begin{pmatrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & - & & \\ + & - & + & - & & & \\ - & + & - & \ddots & & & \\ + & - & & & \ddots & & \\ - & + & & & & & \\ \vdots & & & & & & \end{pmatrix}$$

Utilize zeros!

$$\det\begin{pmatrix} 1 & 0 & 1 & 2 \\ 4 & 1 & 3 & \cancel{0} \\ 4 & 2 & 2 & \cancel{0} \\ 5 & 0 & 0 & 3 \end{pmatrix}$$

"expand about the 4th column"

$$= (-1)^{1+4}\, 2 \cdot \det\begin{pmatrix} 4 & 1 & 3 \\ 4 & 2 & 3 \\ 5 & 0 & 0 \end{pmatrix} + \cancel{0\cdot(\ldots)} + \cancel{0(\ldots)}$$

$$+ (-1)^{4+4} \cdot 3 \cdot \det\begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 3 \\ 4 & 2 & 2 \end{pmatrix} = -2\cdot(-20) + 3\cdot 5$$

$$= 40 + 15$$

$$= 55$$

expand about 3rd row

$$\det\begin{pmatrix} 4 & 1 & 3 \\ 4 & 2 & 3 \\ 5 & 0 & 0 \end{pmatrix} = (-1)^{3+1} \cdot 5 \cdot \det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} + \cancel{0(\ldots)} + \cancel{0(\ldots)}$$

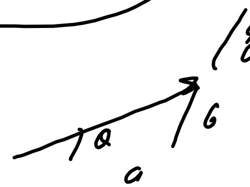
$$= 5\cdot(2-6) = -20$$

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 3 \\ 4 & 2 & 2 \end{pmatrix} = (-1)^{1+1}\cdot 1 \cdot \det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} + \cancel{(0)(\ldots)} + (-1)^{1+3}\cdot 1 \cdot \det\begin{pmatrix} 4 & 1 \\ 4 & 2 \end{pmatrix}$$

expand about first row

$$= (-4) + \det\begin{pmatrix} 4 & 1 \\ 4 & 2 \end{pmatrix} = -4 + 4 = 5$$

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \longleftrightarrow a+bi$



$\sqrt{a^2+b^2}$ scaling

rot: $\arctan\left(\frac{b}{a}\right)$

cols are perp.

$\begin{pmatrix} 100 & -100 \\ 100 & 100 \end{pmatrix}$

length pres $\Longleftrightarrow$ cols perp + unit lengths

$\parallel$
orthogonal — orthonormal basis

symmetric: $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\Big\}$ 3

skew-sym: $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ + 1

2x2 matrices: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 4

sym $\cap$ skew sym $= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

A $n \times n$ matrix

$A = \underset{\text{symmetric}}{S} + \underset{\text{skew symmetric}}{T}$ unique

$= S' + T' \Rightarrow S = S';\ T = T'$

$S = \frac{1}{2}(A + A^t)$, $S^t = \frac{1}{2}(A^t + \underbrace{A^{tt}}_{A})$

$T = \frac{1}{2}(A - A^t)$, $T^t = \frac{1}{2}(A^t - A) = -T$

$$\begin{tikzcd}
\boxed{M_{n\times n}} \arrow[r, "\text{proj}"] \arrow[dr, "\text{proj}"'] & \underline{Sym_n} \\
\mathbb{R}^{n^2} & \underline{Skew\ Sym_n}
\end{tikzcd}$$

$A \circ B = tr(AB^t)$

$\underline{U}$ subspaces of $\mathbb{R}^n$, closed under lin combo.

notion of lin combo

$\underline{M_{n\times n}} \longrightarrow \underline{M_{n\times n}}$

$A \longmapsto A^T$

$$\left.\begin{aligned} (A+B)^T &= A^T + B^T \\ (cA)^T &= c(A^T) \end{aligned}\right\}$$ linear transformation

$P_2 = \{\text{polynomials of deg} \leq 2\}$

$\{a_0 + a_1 x + a_2 x^3\}$

$\underbrace{\{1, x, x^2\}}_{\text{basis}}$

P_2

$D : P_2 \to P_2$

$p \mapsto p'$

$1 \mapsto 0$

$x \mapsto 1$

$x^2 \mapsto 2x$

$a_0 + a_1 x + a_2 x^2$

$$\overset{A}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}} \overset{a}{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}} = \begin{pmatrix} a_1 \\ 2a_2 \\ 0 \end{pmatrix} \leftrightarrow a_1 + 2a_2 x + 0x^2$$

$\mathrm{im}(A) = \mathrm{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}\right)$

is ~ plane in $\mathbb{R}^3$

$\mathrm{Ker}(D)$

$a_1 = a_2 = 0$

$Dp = p'$

$p' = 0 \Rightarrow p$ constant

$\mathrm{im}(D) =$ deg ≤ 1 polynomials

$p \cdot q = \int_0^1 p(x) q(x)\, dx$

Connection w/ transpose + orthogonality

A $n \times m$ matrix

A^t $m \times n$ matrix

$(A^t)_{ij} = A_{ji}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$\vec{v} \cdot \vec{w} = \underline{v_1 w_1 + v_2 w_2 + \dots + v_n w_n}$

$\vec{v}$ $\vec{w}$ $= \underline{\|\vec{v}\| \|\vec{w}\| \cos(\theta)}$

$n \times 1$ $n \times 1$

$\underset{1 \times n}{\vec{v}^t}$ $\underline{(v_1 \ v_2 \ \dots \ v_n)}$

$(1 \times n)(n \times 1) = 1 \times 1$

$\vec{v}^t \vec{w} = (v_1 \ v_2 \ \dots \ v_n) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = v_1 w_1 + \dots v_n w_n$

$= \vec{v} \cdot \vec{w}$

$$\vec{v}^t \vec{w} = \vec{v} \cdot \vec{w}$$

A orthogonal $\iff \|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x}$

$\Updownarrow \qquad\qquad \Updownarrow (?)$

cols A are ONB

$$A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$$

$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\|\vec{v}_i\| = 1$

all perpendicular

$A^{-1} = A^t$

$$A^t A = \begin{pmatrix} - \vec{v}_1^{\,t} - \\ \vdots \\ - \vec{v}_n^{\,t} - \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$$

$(A^t A)_{11} = \vec{v}_1^{\,t} \vec{v}_1 = \vec{v}_1 \cdot \vec{v}_1 = 1$

$(A^t A)_{ij} = \vec{v}_i^{\,t} \vec{v}_j = \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i = j \\ 0 & \text{o/w} \end{cases}$

fix $\vec{v}$ in $\mathbb{R}^3$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^1$

$T(\vec{w}) = \vec{v} \cdot \vec{w}$

matrix of T? 1×3

$\vec{v}^{\,t}$

$T(\vec{w}) = \vec{v}^{\,t}\vec{w}$

$\boxed{\vec{v}^{\,t} \text{ is the matrix of } \vec{v}\cdot(-),\ T}$

$L(\mathbb{R}^3, \mathbb{R}^1) = M_{1\times 3}(\mathbb{R})$

$\mathbb{R}^3 \longrightarrow M_{1\times 3}(\mathbb{R})$

$\vec{v} \longmapsto \vec{v}^{\,t}$

$$(\underline{A}^t \underline{A})_{ij} = (i^{th} \text{ col of } A) \cdot (j^{th} \text{ col of } A)$$

$$(BA)_{ij} = (i^{th} \text{ row of } B) \cdot (j^{th} \text{ col of } A)$$

$$\begin{pmatrix} | & & | \\ v_1 & - - & v_n \\ | & & | \end{pmatrix}$$

ONB $\iff \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$\iff (A^t A)_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$\iff A^t A = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix}$

$\iff A^t = A^{-1}$

$$AA^t = I$$

$$\underset{n \times n}{A} \underset{n \times n}{B} = I_n$$

$\Downarrow$ rank nullity

$$BA = I_n$$

"adjoint property"

$$(A\vec{v})\cdot\vec{w} = \underset{1\times n}{(A\vec{v})^t}\ \underset{n\times 1}{\vec{w}}$$

$$(AB)^t = B^t A^t$$

$$= \underbrace{\vec{v}^t}_{1\times n}\ \underbrace{A^t\vec{w}}_{n\times 1}$$

$$= \vec{v}^t(A^t\vec{w})$$

$$\|$$

$$(A\vec{v})\cdot\vec{w} = \underline{\vec{v}}\cdot(A^t\vec{w})$$

$$T(\vec{v}) = A\vec{v}$$

$$A^t$$

$$T(\vec{v})\cdot\vec{w} = \vec{v}\cdot T^t(\vec{w})$$

def of T^t

adjoint of T

"basis independent"

$$\vec{v}\cdot\vec{w} = \langle\vec{v},\vec{w}\rangle$$

$$\langle A\vec{v},\vec{w}\rangle = \langle\vec{v}, A^t\vec{w}\rangle$$

symmetric

$$\underline{A = A^t}$$

$a_{ij} = a_{ji}$

spectral thm. $\underline{A \text{ symmetric}}$ (*)

$\Updownarrow$

$\exists\, O$ orthogonal s.t. OAO^t diagonal

$$(A\vec{v})\cdot\vec{w} = \vec{v}\cdot(A^t\vec{w})$$

$$= \vec{v}\cdot(A\vec{w})$$

$\{1, 3, 35\}$

$A^t = A^{-1}$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{v}_1 \qquad \vec{u}_1$

~~orthogonal~~

$$A^t A \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$?

s.t. $T(\vec{v}_1) = \vec{u}_1$

$T(\vec{x}) = A\vec{x}$

9 numbers

$T(\vec{e}_i)$ i^{th} col of A

to define T

$\Updownarrow$

choosing the values of $T(\vec{e}_i)$

$\Updownarrow$

cols of A

Given any $\boxed{\vec{w_1}, \vec{w_2}, \vec{w_3}}$ in $\mathbb{R}^3$

there is a unique lin. tran. $\boxed{T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3}$

such that $T(\vec{e_i}) = \vec{w_i}$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = aT(\vec{e_1}) + bT(\vec{e_2}) + cT(\vec{e_3})$$

$$T(\vec{e_i}) = ?$$

$$T\begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\mathbb{R}^n$

$\vec{e_1}, \dots, \vec{e_n}$ notation

warning!

$T(\vec{e_1})$, $T(\vec{e_2})$, $T(\vec{e_3})$ ~~change~~ ??

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_\beta = a\vec{v_1} + b\vec{v_2} + c\vec{v_3}$$

$$\frac{2}{3}\underbrace{T(\vec{e_1})}_{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} + \frac{2}{3}\underbrace{T(\vec{e_2})}_{\begin{pmatrix} d \\ e \\ f \end{pmatrix}} + \frac{1}{3}\underbrace{T(\vec{e_3})}_{\begin{pmatrix} g \\ h \\ i \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $(\vec{v_1}, \vec{v_2}, \vec{v_3}) = \beta$ be a basis for $\mathbb{R}^3$

Given $\vec{w_1}, \vec{w_2}, \vec{w_3}$ in $\mathbb{R}^3$

$\vec{v}$ in $\mathbb{R}^3$

$\vec{v} = a\underline{\vec{v_1}} + b\underline{\vec{v_2}} + c\underline{\vec{v_3}}$ uniquely

there is a unique $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

s.t. $T(\vec{v_i}) = \vec{w_i}$

$$? \quad \underbrace{T\begin{bmatrix} a \\ b \\ c \end{bmatrix}_\beta} = T(a\vec{v_1} + b\vec{v_2} + c\vec{v_3}) = aT(\vec{v_1}) + bT(\vec{v_2}) + cT(\vec{v_3})$$

$T(\vec{v})$

<u>data</u>

$T \; C \longrightarrow \vec{u_1}, \vec{u_2}, \vec{u_3}$ where the basis is sext

and a <u>basis</u>

$\vec{u_i}$ standard basis

$$\left\{ \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}, \vec{0}, \vec{0} \right\}$$

$$\left\{ \vec{0}, \vec{0}, \vec{??} \right\}$$

$$a\vec{0} + b\vec{0} + c\vec{??} = \vec{0}$$

$c = 0$

a, b free

$T \longleftrightarrow$ basis $\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3$

outputs $\vec{w}_1 \quad \vec{w}_2 \quad \vec{w}_3$

$$\begin{pmatrix} 2() \\ 2() \\ 1() \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 2() \\ 2() \\ 1() \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2() \\ 1() \\ 1() \end{pmatrix} = T^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \cancel{\quad} & 2() \\ & 2() \\ & 1() \end{pmatrix}$$

$\|$

A^t

A^{-1} a thing, A orthog

A^{-1} repr T^{-1}

$\vec{T}_1$ $\quad T(\vec{v}) = A\vec{v}$

$$\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \longmapsto \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

lin. indep.

$$\underbrace{(\overbrace{(v_1, \dots, v_k)}^{\text{lin. indep.}}, \dots, v_n)}_{\text{basis}}$$

"extending to a basis"

$$\vec{v}_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

Let $\vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$ s.t. $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis

$\vec{v}_1 \neq \vec{0}$

e.g.

$\vec{v}_1$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

ONB

$\updownarrow$

cols of orthogonal matrix

countless, arbitrary

$\vec{v}_1 \longmapsto \vec{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\vec{v}_2 \longmapsto ??\ \vec{w}_2$

$\vec{v}_3 \longmapsto ??\ \vec{w}_3$

$\vec{0}$

$\vec{0}$

goal: find $\underbrace{\vec{v}_2, \vec{v}_3}_{\text{input}}$, $\underbrace{\vec{w}_2, w_3}_{\text{output}}$ in $\mathbb{R}^3$

s.t.

- $\vec{v}_1, \vec{v}_2, \vec{v}_3$ basis
- The lin. tran. T s.t. $T(\vec{v}_i) = \vec{w}_i$ is orthogonal

$$T\begin{pmatrix}2/3\\2/3\\1/3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

length 1   length 1

$$\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

length 1

$\begin{pmatrix}2/3\\2/3\\1/3\end{pmatrix}$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ ONB

$\vec{w}_1, \vec{w}_2, \vec{w}_3$ ONB

$\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\vec{v}_i \mapsto \vec{w}_i$

$\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$

Given $\vec{v_1}, \vec{v_2}, \vec{w_3}$ ONB

any $\vec{w_1}, \vec{w_2}, \vec{w_3}$ ONB

there is a unique $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

s.t. $T(\vec{v_i}) = \vec{w_i}$

and T is orthogonal

$\vec{v_1} = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}, \quad \vec{w_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

find

$\vec{v_2}$ $\qquad \vec{w_2}$

$\vec{v_3}$ $\qquad \vec{w_3}$

s.t.

$\underbrace{\vec{v_1}, \vec{v_2}, \vec{v_3}}$ $\qquad \vec{w_1}, \vec{w_2}, \vec{w_3}$ ONB

find this somehow

- extend to a basis
- Gram Schmidt

$\vec{w_1} = \vec{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

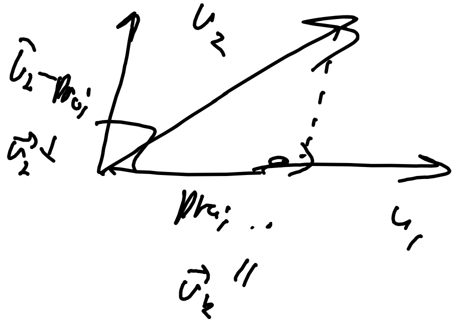
$\vec{w_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{w_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$v_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad v_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{array}{ccc} & & \\ \left(\begin{matrix} 2/3 & 1 & 0 \\ 2/3 & 0 & 1 \\ 1/3 & 0 & 0 \end{matrix}\right) & & \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \qquad \det = \det\begin{pmatrix} 2/3 & 1 \\ 1/3 & 0 \end{pmatrix} \neq 0$$

v_1, v_2, v_3 basis

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{v}_1$$



$$\vec{u}_2 = \frac{\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1}{\|\vec{v}_2 - (\vec{v}_2 \cdot u_1)\vec{u}_1\|}$$

$$\frac{\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}2/3\\2/3\\1/3\end{pmatrix}}{\|\ldots\|}$$

$$\begin{pmatrix}a\\b\end{pmatrix}, \quad \begin{pmatrix}-b\\a\end{pmatrix}$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix}$$