Def.

V a subspace of $\mathbb{R}^n$

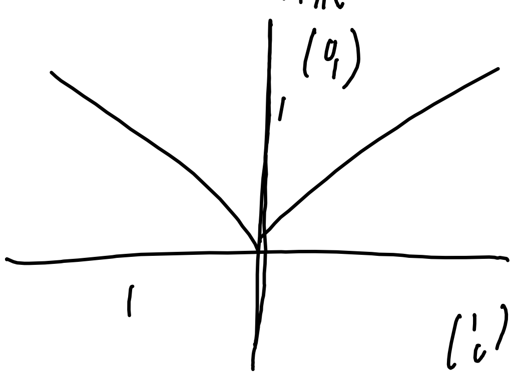
orthogonal/perpendicular complement

$$V^\perp = \{\vec{w} \text{ in } \mathbb{R}^n \text{ such that } \vec{v}\cdot\vec{w}=0 \text{ for all } \vec{v} \text{ in } V\}$$

$\vec{v}\cdot\vec{w}=0 \iff \vec{v}, \vec{w}$ perp.

($\vec{0}$ perp. to all)

$V^\perp$ = {vectors in $\mathbb{R}^n$ which are perp to all vectors in V}

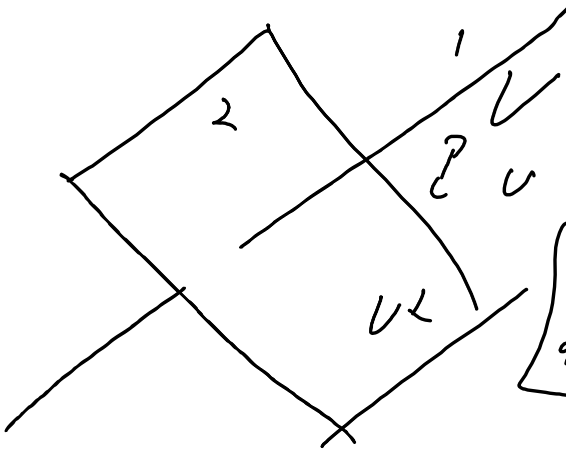


$\begin{pmatrix}1\\0\end{pmatrix}$ in $\mathbb{R}^2$

$\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}=x$

$V=\text{span}\begin{pmatrix}1\\0\end{pmatrix}$. $V^\perp=\left\{\begin{pmatrix}x\\y\end{pmatrix} \text{ s.t. } x=0\right\}$

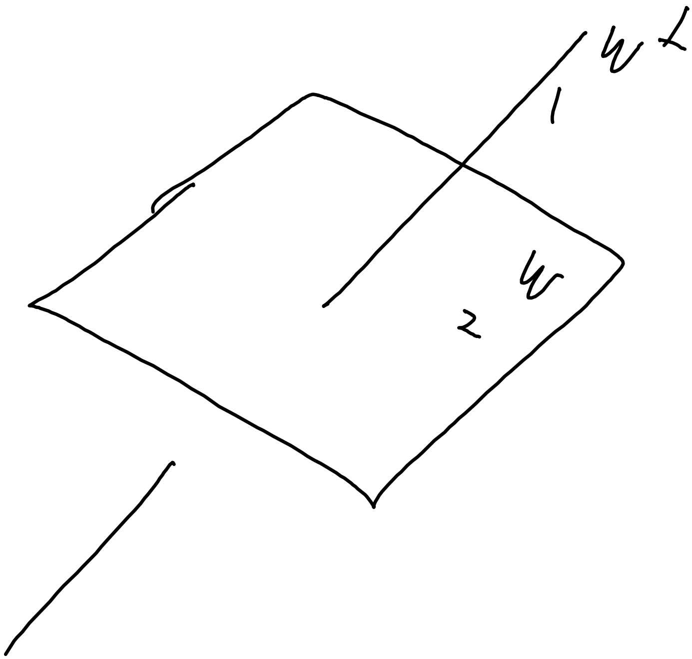
$=\text{span}\begin{pmatrix}0\\1\end{pmatrix}$ y-axis



$\begin{pmatrix}1\\1\\1\end{pmatrix}$ $V = \text{span}\begin{pmatrix}1\\1\\1\end{pmatrix}$

? V and $V^\perp$

$\left\{\begin{array}{l}\text{all in } V \text{ perp } V^\perp \\ \text{all in } V^\perp \text{ perp } V\end{array}\right\}$



$$\underline{(V^\perp)^\perp} = V$$

V, $V^\perp$ subspaces $\mathbb{R}^n$

$$\dim V + \dim V^\perp = n$$

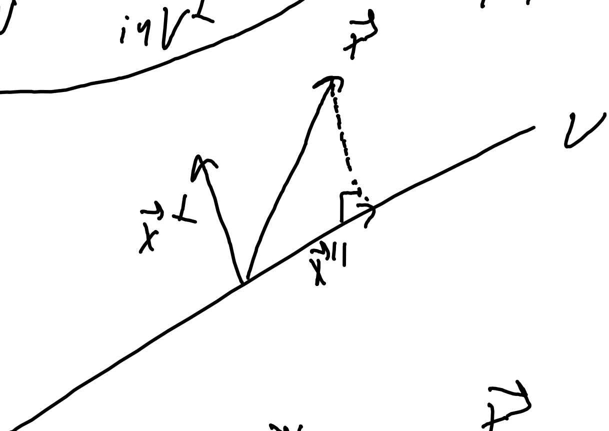
Projection.

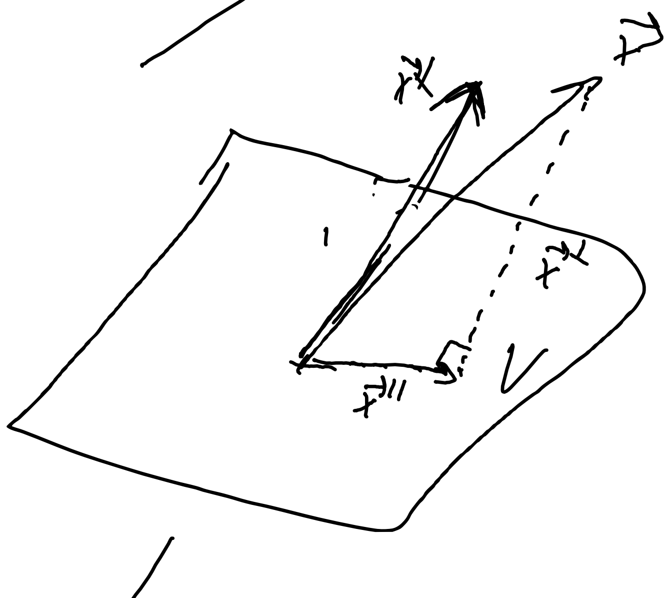
V subspace of $\mathbb{R}^n$

$\vec{x}$ in $\mathbb{R}^n$

$$\vec{x} = \underset{\text{in } V}{\vec{x}^{\parallel}} + \underset{\text{in } V^{\perp}}{\vec{x}^{\perp}}$$

Claim! unique!





$\text{proj}_V(\vec{x}) \overset{\text{def}}{=} \vec{x}^{\parallel}$

orthogonal projection

$\mathbb{R}^n$

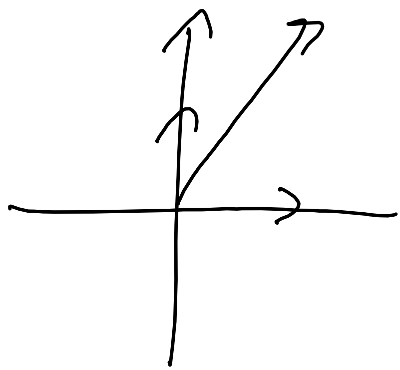
$\mathbb{R}^4$

$\vec{v}, \vec{w}$ in $\mathbb{R}^4$

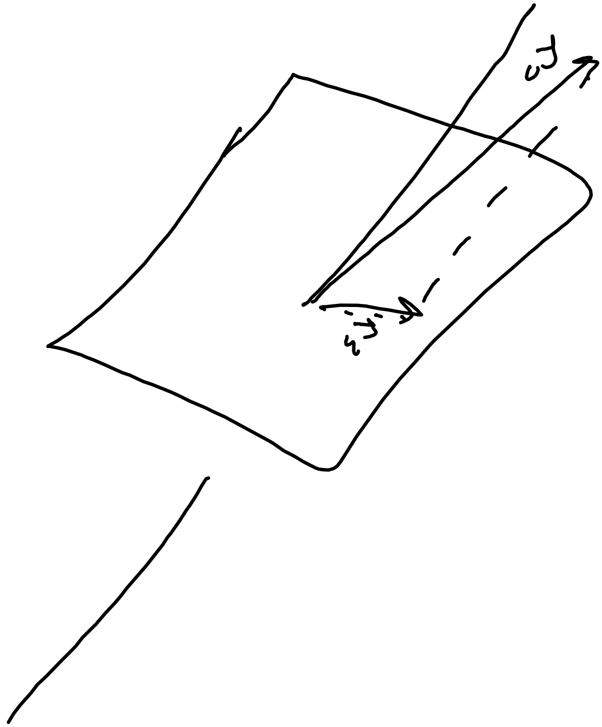
$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\mathbb{R}^3 \times \mathbb{R}$

$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, t\right)$



$$\text{span}(\vec{v}, \vec{w}) = \begin{cases} 0 \text{ dim} & \vec{v} = \vec{w} = \vec{0} \quad \text{pt, origin} \\ 1 \text{ dim} & \text{one nonzero, lin. dep} \quad \text{line} \\ 2 \text{ dim} & \text{lin indep} \quad \text{plane} \end{cases}$$



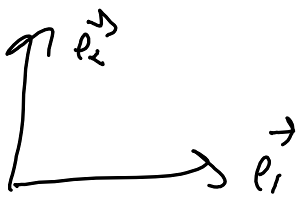
$\vec{x} = \underline{\vec{x}^{\parallel}} + \underline{\vec{x}^{\perp}}$

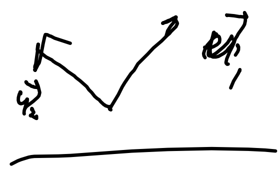
$\mathbb{R}^n$, ONB $\vec{u}_1, \dots, \vec{u}_n$

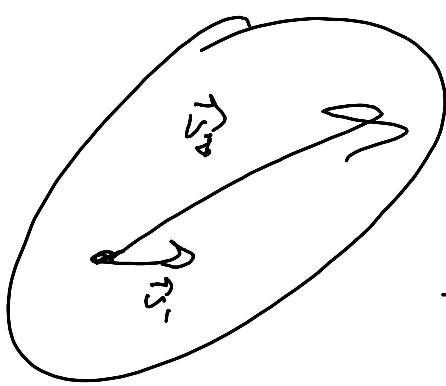
s.t. $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i=j \\ 0 & o/w \end{cases}$

$\|\vec{u}_i\| = 1$

all perp to each other







not ONB

is basis

V subspace of $\mathbb{R}^n$

$\vec{u}_1, \dots, \vec{u}_k$ ONB of V

$, \vec{u}_{k+1}, \dots, \vec{u}_n$

$$\underbrace{\vec{u}_1 \dots \vec{u}_k}_{\text{ONB } V}, \underbrace{\vec{u}_{k+1} \dots, \vec{u}_n}_{\text{ONB } V^{\perp}} \quad \text{ONB } \mathbb{R}^n$$

$$O = \begin{pmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{pmatrix}$$

$$O^T = \begin{pmatrix} - & \vec{u}_1 & - \\ & \vdots & \\ - & \vec{u}_n & - \end{pmatrix} = O^{-1}$$

orthogonal

$\vec{x}$ in $\mathbb{R}^n$

$$\vec{x} = x_1 \vec{u}_1 + \dots + x_n \vec{u}_n$$

$$\vec{x} = x_1 \vec{e}_1 + \dots + x_n \vec{e}_n$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$1 \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2$$

$$\vec{x} \cdot \vec{u}_1 = x_1 \underbrace{\vec{u}_1 \cdot \vec{u}_1}_{1} + x_2 \cancelto{0}{\vec{u}_2 \cdot \vec{u}_1} \cancelto{0}{+ \dots +} x_n \cancelto{0}{\vec{u}_n \cdot \vec{u}_1}$$

$$= x_1$$

$$x_i = \vec{x} \cdot \vec{u_i}$$

$$\vec{x} = \underbrace{\underbrace{(\vec{x} \cdot \vec{u_1})}\vec{u_1}}_{\text{proj}_{\vec{u_1}}(\vec{x})} + \cdots + \underbrace{\underbrace{(\vec{x} \cdot \vec{u_n})}\vec{u_n}}_{\text{proj}_{\vec{u_n}}(\vec{x})}$$

$$\vec{x} = \underbrace{(\vec{x} \cdot \vec{u_1})\vec{u_1} + \cdots + (\vec{x} \cdot \vec{u_k})\vec{u_k}}_{\parallel \text{ def } \ \vec{x}^{\parallel}} + \underbrace{(\vec{x} \cdot \vec{u_{k+1}})\vec{u_{k+1}} + \cdots + (\vec{x} \cdot \vec{u_n})\vec{u_n}}_{\parallel \text{ def } \ \vec{x}^{\perp}}$$

$\vec{u_1}, \ldots, \vec{u_k}$ in V

$\vec{x}^{\parallel}$ in V

$\underbrace{\text{span}(\underbrace{\vec{u_{k+1}}, \ldots, \vec{u_n}})}\cap V = \{\vec{0}\}$

$$\text{proj}_V(\vec{x}) = \sum_{i=1}^{n} (\vec{x} \cdot \vec{u_i})\vec{u_i}$$

$\text{im}(\text{proj}_V) = V$

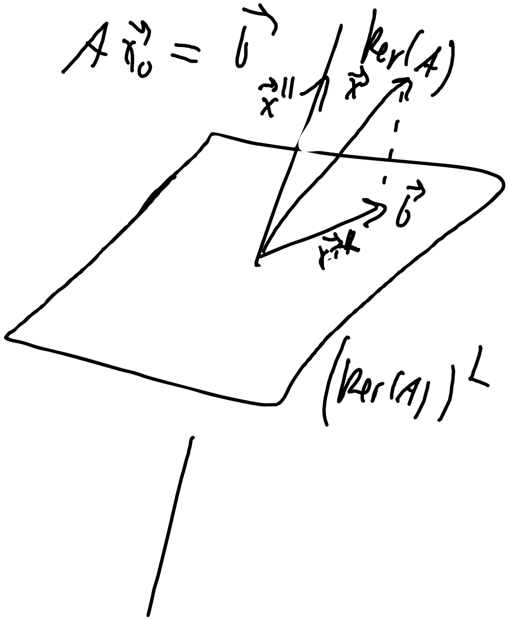
$\ker(\text{proj}_V) = V^{\perp}$

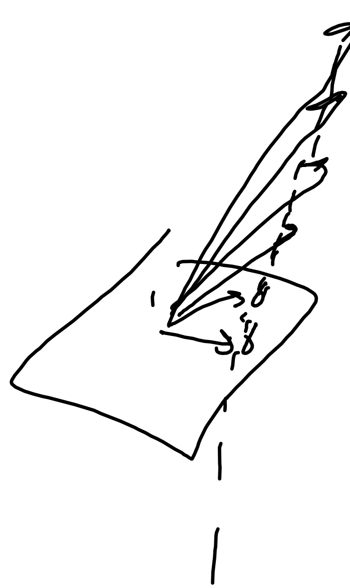
rank nullity

2. $\underline{A\vec{x} = \vec{b}}$ consistent system

a) why can find $\underline{\vec{x}_0}$ in $\underline{(\ker(A))^\perp}$
which solves the system?

$A\vec{x}_0 = \vec{b}$





Let $\vec{x}$ be a solution

$A\vec{y} = \vec{b}$

$\vec{x}$ in $\mathbb{R}^n$ $\rightsquigarrow$ $(\ker(A))^\perp$

$\vec{x}^{\parallel} = \operatorname{proj}_{\ker(A)}(\vec{x})$

$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$

$\ker A$ $\quad$ $(\ker(A))^\perp$

$$\vec{b} = A\vec{x}$$

$$= A\left(\underbrace{\vec{x}^{\parallel}}_{\vec{0}} + \vec{x}^{\perp}\right)$$

$$= \cancel{A\vec{x}^{\parallel}} + A\left(\vec{x}^{\perp}\right)$$

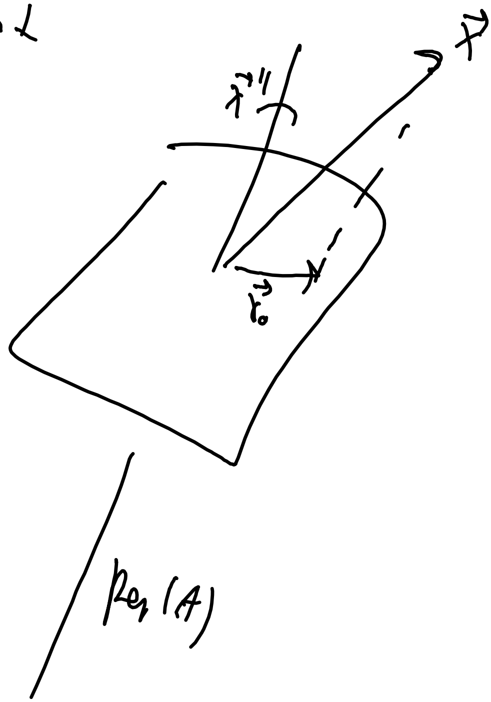
$$\vec{x}^{\parallel} = \ker A$$

$$= \vec{0} + A\left(\vec{x}^{\perp}\right)$$

$$= A\left(\vec{x}^{\perp}\right)$$

$$A\vec{x}^{\perp} = \vec{b}$$

$$\vec{r}_0 = \vec{x}^{\perp}$$



general solution   homogeneous soln

any solution $\vec{X}$ is written $\vec{a}_j \vec{X}_h + \vec{X}_0$ ← particular solution

in ker(A)   in (ker (A))$^\perp$

h <-> homogeneous

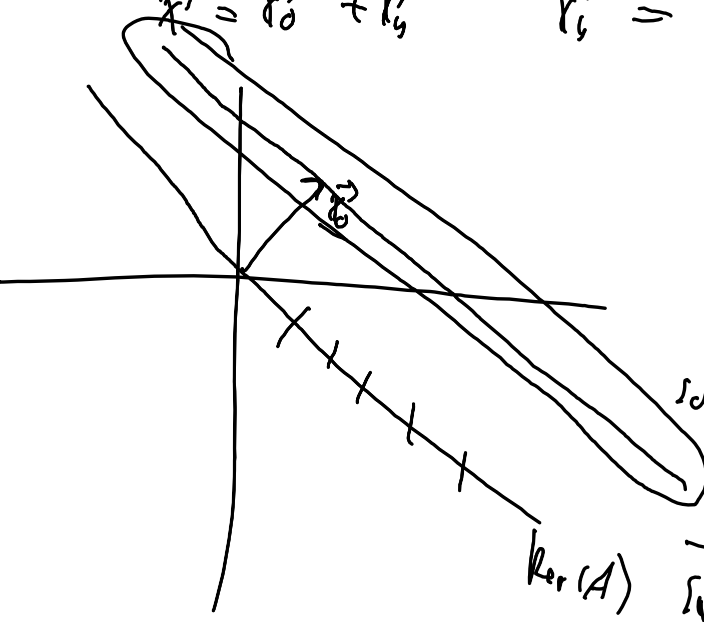
ker(A) <-> solution $A\vec{x} = \vec{0}$

$\vec{x}_0$, $\vec{x}$ both solution

$A(\vec{x}_0 - \vec{x}) = A\vec{x}_0 - A\vec{x} = \vec{b} - \vec{b} = \vec{0}$

$\vec{x}_0 - \vec{x} \in \ker(A)$

$\vec{x} = \vec{x}_0 + \vec{x}_h \qquad \vec{x}_h \overset{\text{check}}{=} \vec{x} - \vec{x}_0$



solns for $A\vec{x} = \vec{b}$ parallel

ker(A)   solns to $A\vec{x} = \vec{0}$ parallel

Midterm T/F A 2×2 I_2 2×2

A^2 2×2

$A^2 = I_2$ $A = \pm I_2$?

F

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x in $\mathbb{R}$ $x^2 = 1 \Rightarrow x = 1,\ x = -1$

$x(y+z) = xy + xz$

$A = B$

$A - B = 0$

$A(B+C) = AB + AC$

$x^2 = 1$ (characteristic polynomial)

$x^2 - 1 = 0$

$\Updownarrow$

$(x-1)(x+1) = 0$

$x - 1 = 0$ $x = 1$

or

$x + 1 = 0$ $x = -1$

$\therefore x = \pm 1$

$(A - I_2)(A + I_2)$

$A(A + I_2) - I_2(A + I_2)$

$A^2 + AI_2 - I_2 A - I_2^2$

$A^2 + A - A - I_2 = A^2 - I_2$

since $A^2 = I_2$ hypothesis

$A^2 - I_2 = O_2$ $\Downarrow$

$(A - I_2)(A + I_2) = O_2$ $\Downarrow$?

$\Downarrow$

$\begin{cases} A - I_2 = 0 & A = I_2 \\ \text{or} & \\ A + I_2 = 0 & A = -I_2 \end{cases}$

A, B $\underline{n \times m}$ matrices

$$(A+B)_{ij} \overset{\text{def}}{=} A_{ij} + B_{ij}$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{pmatrix}2 & 2\\ 3 & 5\end{pmatrix}$$

$$(cA)_{ij} \overset{\text{def}}{=} c(A_{ij})$$

$$2\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{pmatrix}2 & 4\\ 6 & 8\end{pmatrix}$$

$\underset{\underline{n\times m}}{A}\ \underset{\underline{m\times k}}{B}$ — $I_n \underset{n\times m}{A} = A$

— $\underset{n\times m}{B} I_m = B$

— $(\underset{n\times m}{A} + \underset{n\times m}{B})\underset{m\times k}{C} = AC + BC$

— $\underset{n\times m}{A}(\underset{m\times k}{B} + \underset{m\times k}{C}) = AB + AC$

— $c(AB) = (cA)B = A(cB)$

$$\underbrace{AB}_{n\times n \;\; n\times n} \neq BA$$

$$AB = O \not\Rightarrow A = O \text{ or } B = O$$

$$A^2 = I_2$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\neq -I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$A - I_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \neq O$$

$$A + I_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq O$$

$$\underbrace{(A - I_2)}_{\neq O}\underbrace{(A + I_2)}_{\neq O} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$