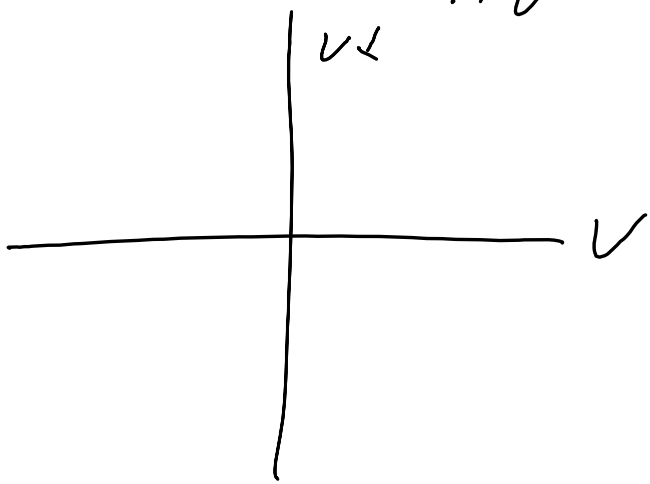
Definitions

1. V a subspace of $\mathbb{R}^n$

$$V^\perp \overset{(\text{def})}{=} \{ \vec{w} \text{ in } \mathbb{R}^n \text{ s.t. } \underbrace{\vec{v}\cdot\vec{w}=0}_{\vec{v}\perp\vec{w}} \text{ for all } \vec{v} \text{ in } V \}$$

perp. to all vectors in V

$\mathbb{R}^2$

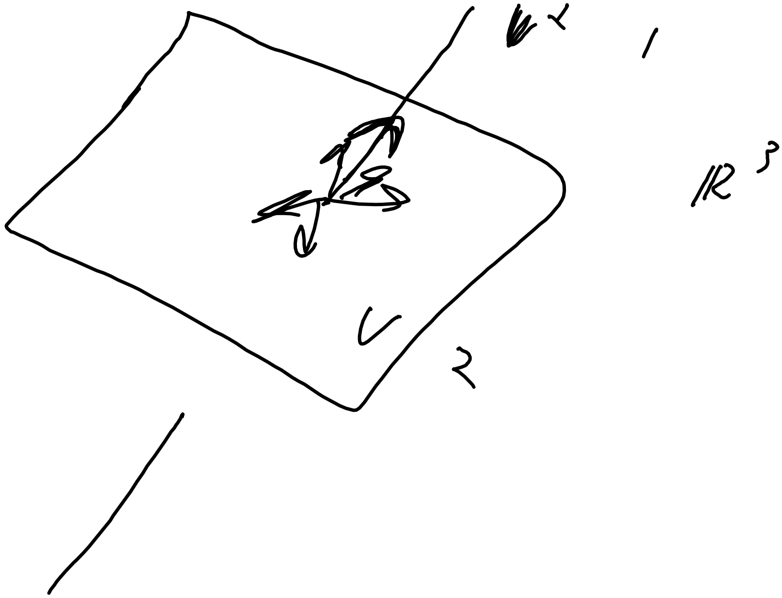


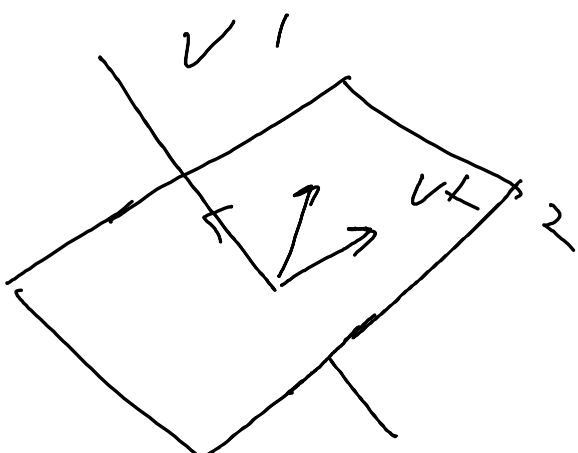
$V = \text{span}\begin{pmatrix}1\\0\end{pmatrix}$

$V^\perp$, $\begin{pmatrix}x\\y\end{pmatrix}$, $\begin{pmatrix}x\\y\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}=0$

$$\begin{pmatrix}x\\y\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = x\cdot 1 + y\cdot 0 = x$$

$V^\perp = \text{span}\begin{pmatrix}0\\1\end{pmatrix}$





$(V^\perp)^\perp = V$

$\boxed{\dim V + \dim V^\perp = n}$ *

rk-null.

V contained in $(V^\perp)^\perp$

$\vec{v}$ in V

$\vec{v}$ perp. to everything in $V^\perp$

$\vec{w}$ in $V^\perp$ $\quad \vec{v} \cdot \vec{w} = 0$

$\vec{v} \in (V^\perp)^\perp$

$V \subseteq (V^\perp)^\perp$

same dimension

$$\dim V \overset{(?)}{=} \dim (V^\perp)^\perp$$

$$\dim(V^\perp) = n - \dim V$$

$$\dim \underbrace{(V^\perp)^\perp} = n - \dim V^\perp$$

$$= n - (n - \dim V)$$

$$= \dim V$$

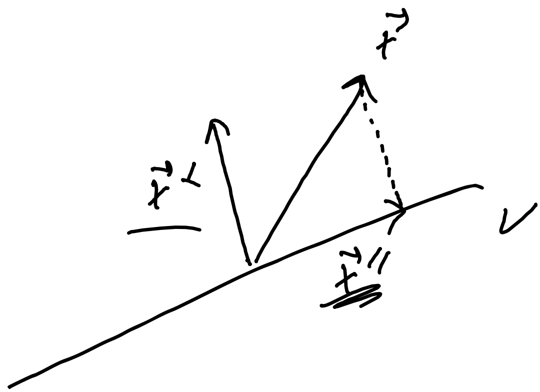
$$V \subseteq (V^\perp)^\perp$$

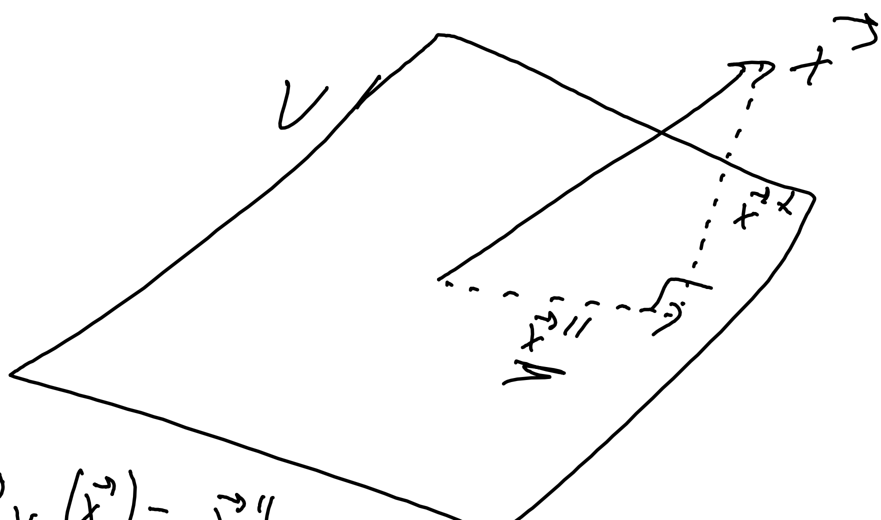
therefore $V = (V^\perp)^\perp$

Orthogonal projection

V subspace of $\mathbb{R}^n$

$\underline{\vec{x}}$ write it $\vec{x} = \underset{\text{in } \underline{V}}{\vec{x}^{\parallel}} + \underset{\text{in } \underline{V^{\perp}}}{\vec{x}^{\perp}}$





$\mathrm{proj}_V(\vec{x}) = \underline{\vec{x}^{\parallel}}$

"orthogonal projection of $\vec{x}$ onto V"

image of proj_V?

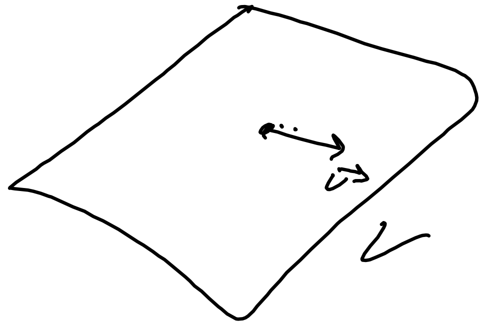
$= V$

contained in V

$\vec{v}$ in V

$\text{proj}_V(\vec{v}) = \vec{v}$

$\vec{v} = \vec{v}^{\parallel}$



Kernel of proj_V?

$\{\vec{0}\}$ in kernel

$V^{\perp}$

$\vec{x} = \vec{x}^{\parallel} + \boxed{\vec{x}^{\perp}}$

$\text{proj}_V(\vec{x}) = \vec{x}^{\parallel}$

$\vec{x}$ in $V^{\perp}$

$\vec{x} = \underset{\text{in } V}{\vec{0}} + \underset{\text{in } V^{\perp}}{\vec{x}}$

$\text{proj}_V(\vec{x}) = \vec{0}$

$V^{\perp}$ contained in $\ker(\text{proj}_V)$

$\text{proj}_V(\vec{x}) = 0$, $\vec{x}^{\parallel} = \vec{0}$

$\vec{x} = \underbrace{\vec{x}^{\parallel}}_{0} + \vec{x}^{\perp}$, ~~so~~ $\vec{x} = \vec{x}^{\perp}$ $\quad \vec{x}$ in $V^{\perp}$

$$\operatorname{im} \operatorname{proj}_U = U$$

$$\ker \operatorname{proj}_U = U^\perp$$

$$\dim(\operatorname{im}(\operatorname{proj}_U)) + \dim(\ker(\operatorname{proj}_U)) = n$$

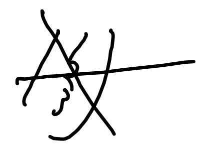
$$\dim U + \dim U^\perp = n$$

$$U = (U^\perp)^\perp$$

2. Consider a consistent system

$$A\vec{x}=\vec{b}$$

a) Show that then is a solution $\boxed{\vec{x}_0}$ which is in $(\ker(A))^{\perp}$



Consistent system? a solution exists

$$A\vec{x}=\vec{b} \quad ?$$

Is there $\vec{x}$ so that $A\vec{x}=\vec{b}$?

Let $\vec{x}$ be a solution to $A\vec{x}=\vec{b}$

in $\mathbb{R}^n$

Looking for something in $(\ker A)^{\perp}$

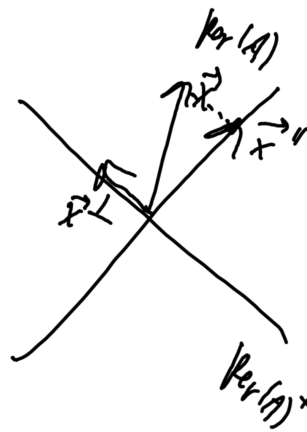
$V=\ker(A)$

$$\vec{x} = \underbrace{\vec{x}^{\parallel}}_{\ker(A)} + \underbrace{\vec{x}^{\perp}}_{(\ker(A))^{\perp}}$$

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

in $\ker(A)$ $\qquad$ $(\ker(A))^{\perp}$

$$A\vec{x} = \vec{b}$$



$$\vec{b} = A\vec{x} = A(\vec{x}^{\parallel} + \vec{x}^{\perp})$$

$$= A\vec{x}^{\parallel} + A\vec{x}^{\perp}$$

0

$\vec{x}^{\parallel}$ in $\ker(A)$

def $\quad A\vec{x}^{\parallel} = \vec{0}$

$$= \vec{0} + A\vec{x}^{\perp}$$

$$= A\vec{x}^{\perp}$$

$$\vec{x}_0 = \vec{x}^{\perp}$$

$\vec{b} = A\vec{x}^{\perp}$ $\quad$ a solution! $\quad$ in $(\ker(A))^{\perp}$

A $n \times m$ matrix

$$\underset{n\times m}{A}\ \underset{m\times 1}{\vec{x}} = \underset{n\times 1}{\vec{b}}$$

$\ker(A)$ subspace of $\mathbb{R}^m$

$\operatorname{im}(A)$ subspace of $\mathbb{R}^n$

$T(\vec{x}) = A\vec{x}$

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^{n}$

m

ker

input

im

output

$\vec{x}$ in $\mathbb{R}^m$

$\ker(A) \subseteq \mathbb{R}^n$

$\cancel{\operatorname{im}(A) \in \mathbb{R}^n}$

$m = 3$

$n = 2$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}$

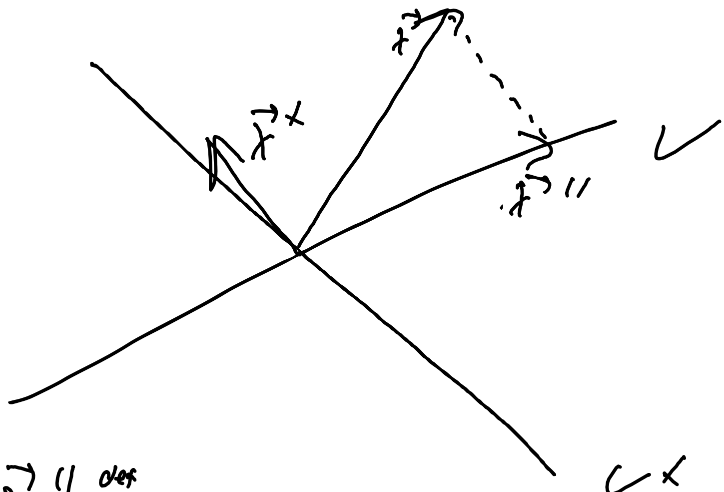
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cancel{\to} \begin{pmatrix} x \\ y \end{pmatrix}$

quiz! $\underline{V}$ subspace of $\mathbb{R}^m$

$\vec{x}$ in $\mathbb{R}^m$ $\quad \underset{\text{unique}}{\underline{\vec{x}}} = \underline{\vec{x}^{\parallel}} + \underline{\vec{x}^{\perp}}$

$\vec{x}^{\parallel}$ in V

$\vec{x}^{\perp}$ $V^{\perp}$



$\vec{x}^{\parallel} \overset{\text{def}}{=} \text{proj}_V(\vec{x})$

$\vec{x}^{\perp} \overset{\text{def}}{=} \vec{x} - \text{proj}_V(\vec{x})$

why can any solution $\vec{x}$

($\text{to } A\vec{x} = \vec{b}$)

be written as $\vec{x} = \vec{x}_h + \vec{x}_0$

$\vec{x}_h$ in $\ker(A)$

$\vec{x}_0$ $\ker(A)^\perp$

h - homogeneous

$$\frac{A\vec{x}_h = \vec{0}}{A\vec{x} = \vec{b}}$$

$\vec{x}$ sol.

$\vec{x}_0$ sol from before, $\vec{x}_0$ in $\ker(A)^\perp$

$$A(\vec{x} - \vec{x}_0) = A\vec{x} - A\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}$$

$\vec{x}$, $\vec{x}_0$ both solutions

$\Updownarrow$

Set of solution is parallel

$\vec{x} - \vec{x}_0$ in $\ker(A)$

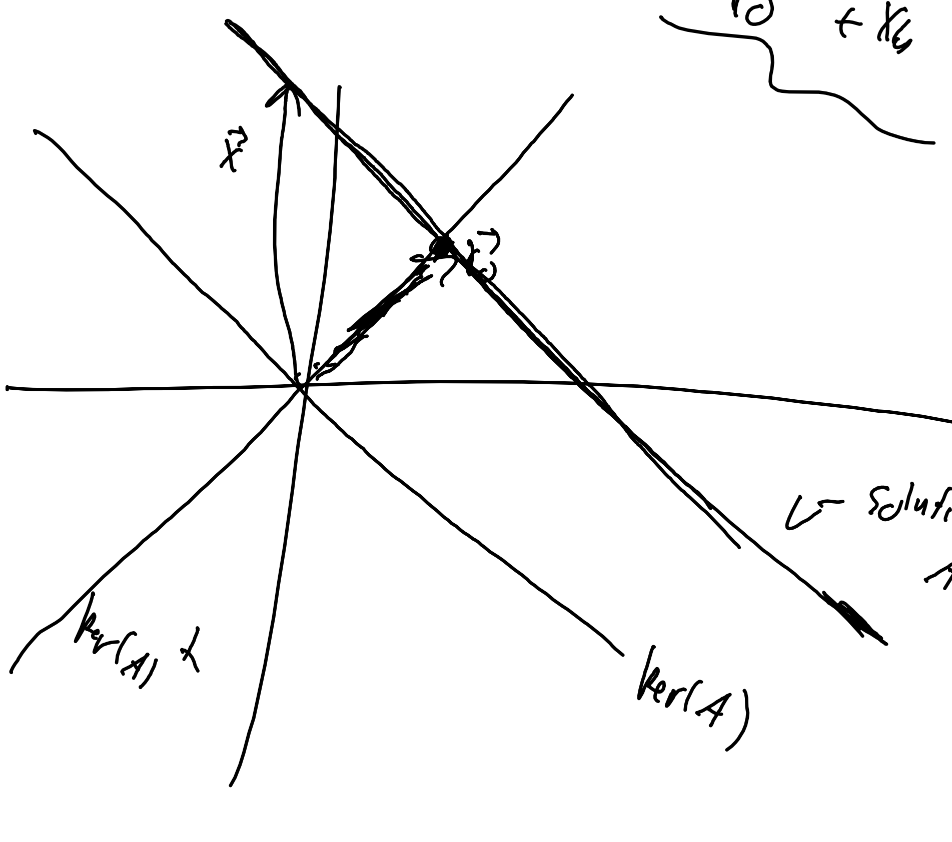
$\vec{x}_h = \vec{x} - \vec{x}_0$

$\vec{x}$ in $\vec{x}_0 + \ker(A)$

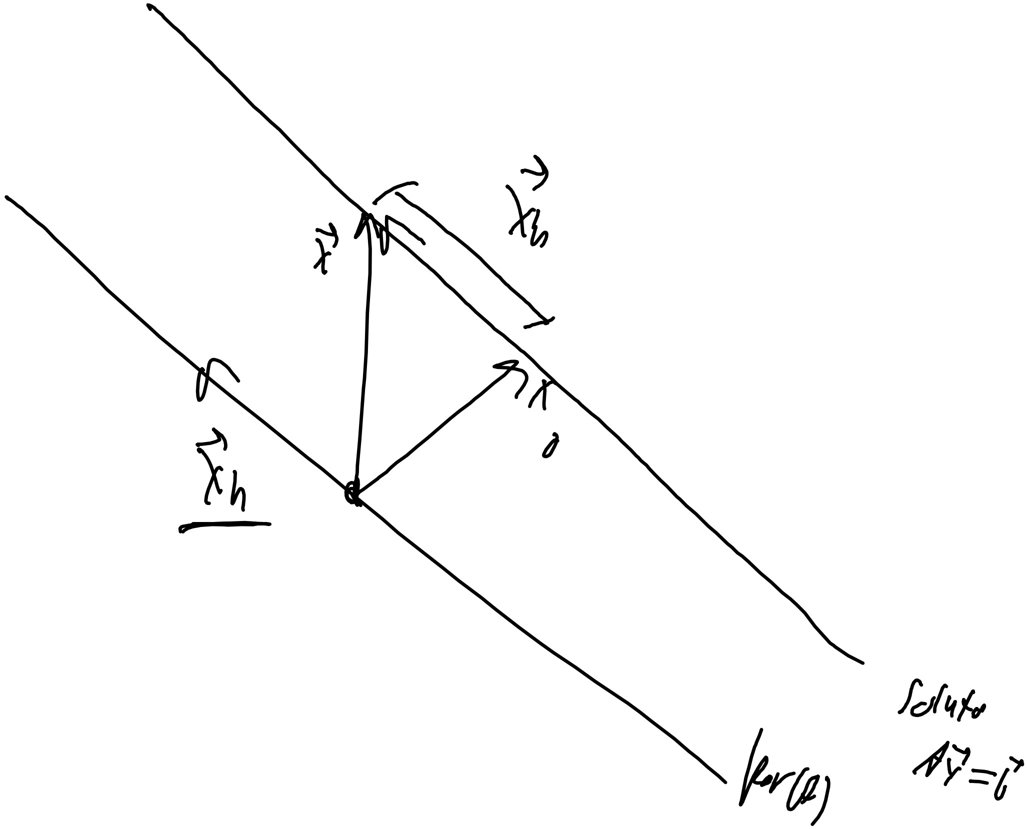
$\vec{x} = \vec{x}_h + \vec{x}_0$

$\vec{x}_0 + \vec{x}_h$

$x_h \in \ker(A)$



Solutions to $A\vec{x} = \vec{b}$!



$\vec{x}$
$\vec{x}_h$
$\vec{x}_0$
$\vec{x}_h$
ker(A)
Soluto
$A\vec{y}=\vec{b}$

$\{1, 1, 3\}$

$\vec{v}$ s.t. $\underline{\sum v_i = 1}$

min'l length.

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 \ldots + v_n^2}$$

$$\cancel{\neq} \quad v_1 + v_2 + \ldots v_n$$

$$\vec{v} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$v_1 + v_2 = \frac{1}{2} + \frac{1}{2} = 1 \qquad \neq$$

$$\sqrt{v_1^2 + v_2^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$\cancel{\sum |v_i|}$ ℓ^1

$\cancel{\sqrt{\sum v_i^2}}$ ℓ^2

$\boxed{n=2}$

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{x+y=1}{y=-x+1}$$

$(x+y)^2 = 1^2 = 1$

$\rightarrow x^2+xy+xy+y^2=1$

??

$x<0$

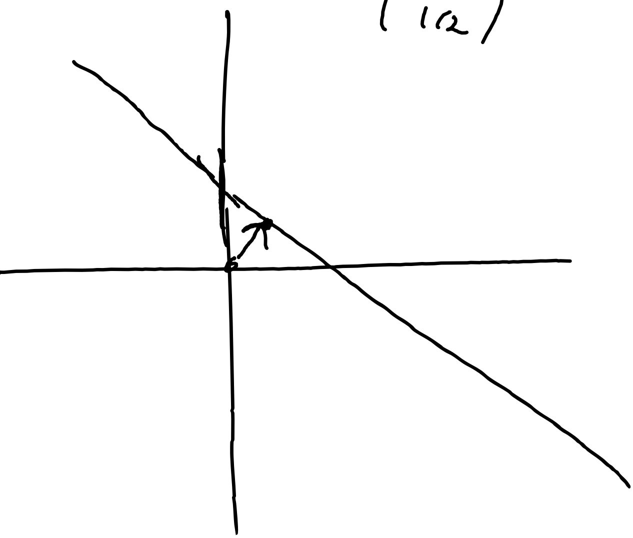
$y>0$

$xy<0$

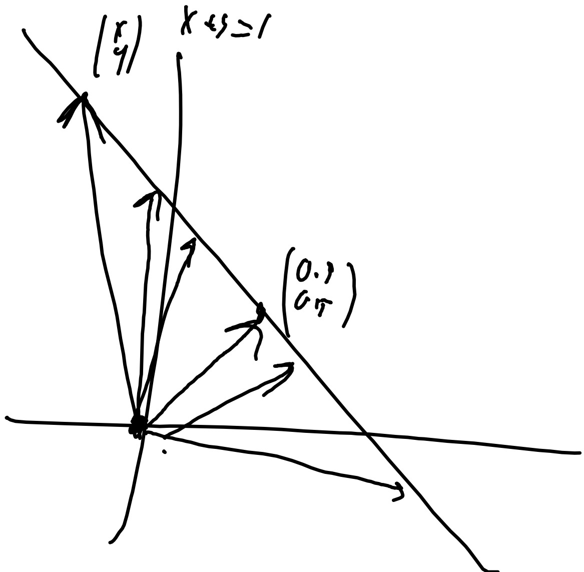
$$1 = \underbrace{(x^2+y^2)}_{=\|u\|^2} + xy$$

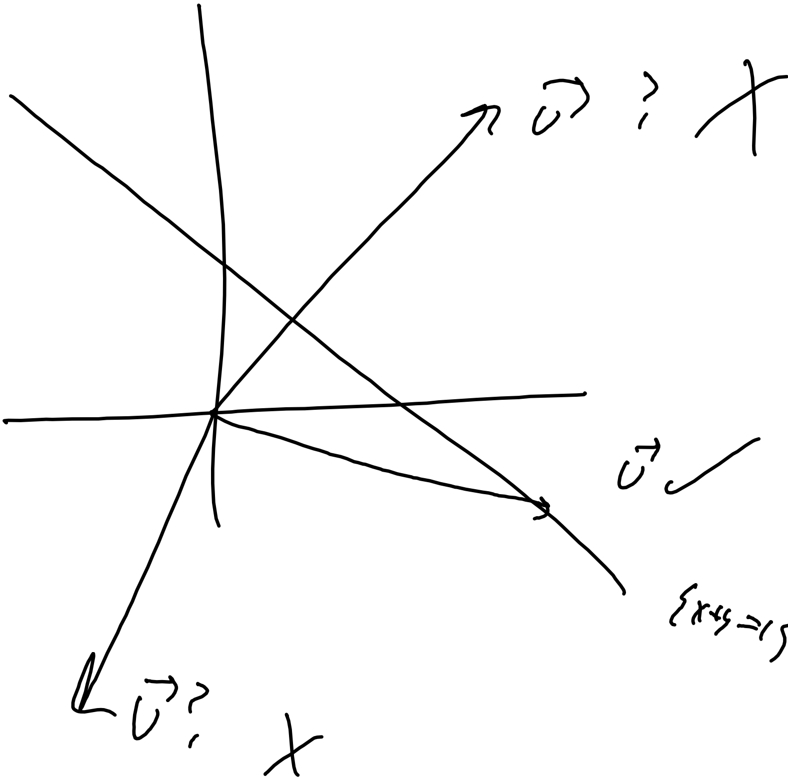
$$\|u\|^2 = \frac{1-xy}{x}$$

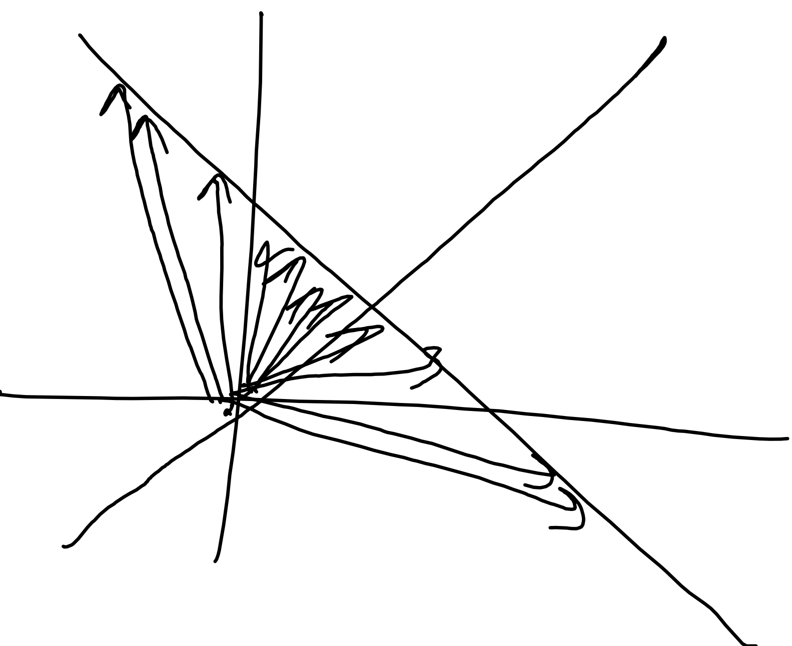
$\begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix}$

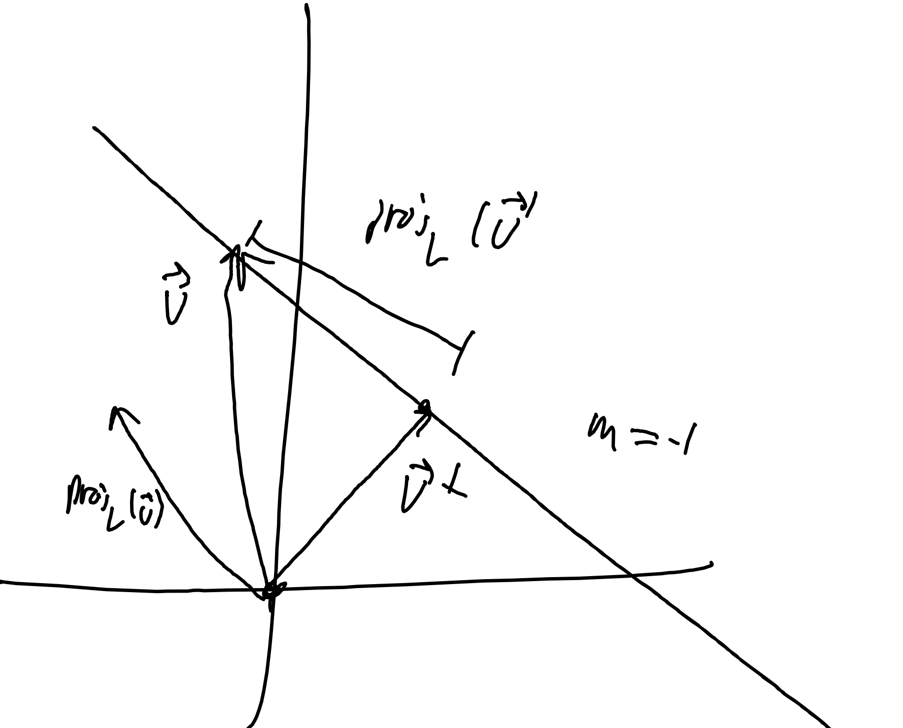


$x+y=1$



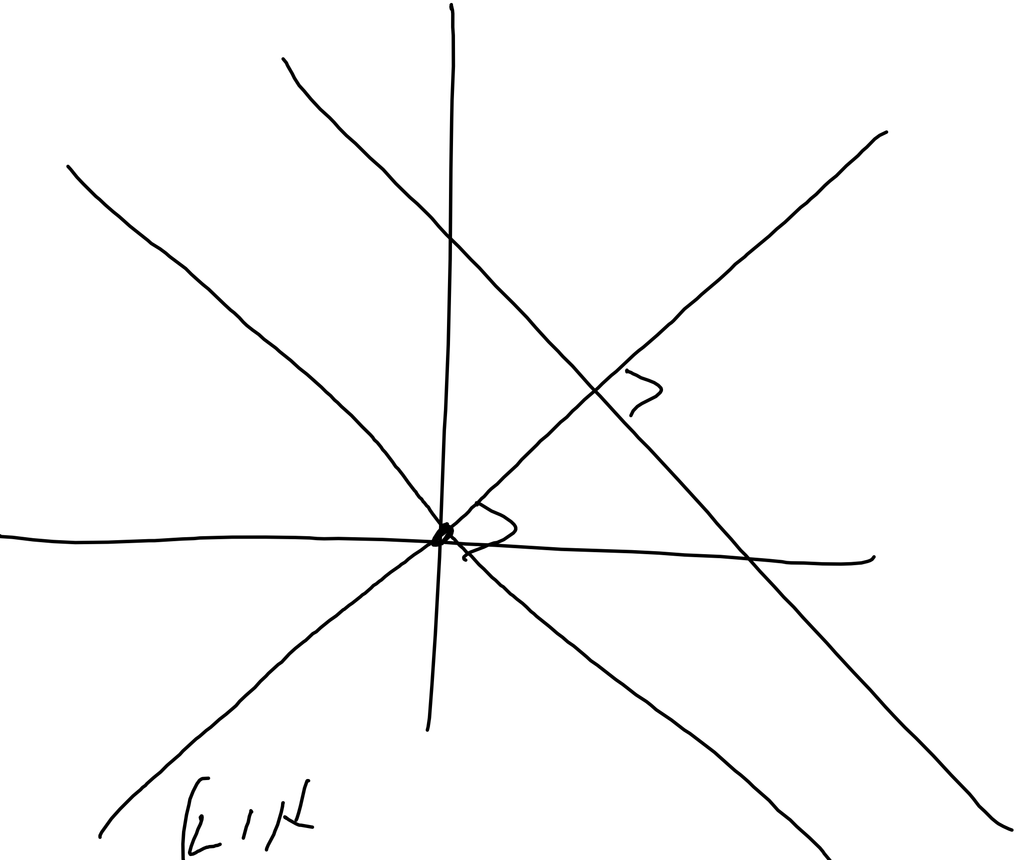

$\vec{v}$? ✗
$\vec{v}$ ✓
$\{x+y=1\}$
$\vec{v}$? ✗





$\|v^\perp\| \le \|\vec{v}\|$

$n = 2 \rightarrow \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$



$(L')^\perp$

L

L'

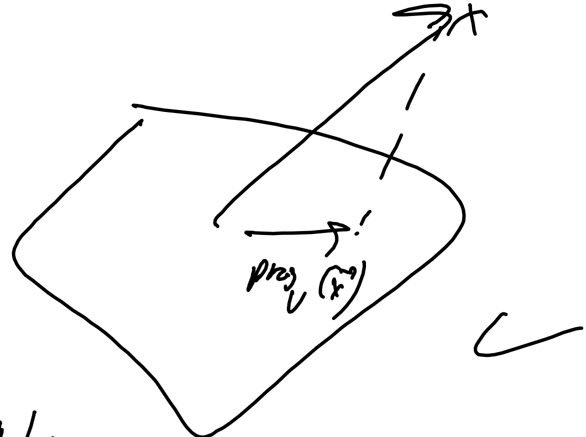
$= \text{span}\begin{pmatrix}1\\-1\end{pmatrix}$

$x+y=1$

$\begin{pmatrix}x\\y\end{pmatrix} \in \begin{pmatrix}1\\-1\end{pmatrix}^\perp$

$\begin{pmatrix}x\\y\end{pmatrix} \cdot \begin{pmatrix}1\\-1\end{pmatrix} = 0$

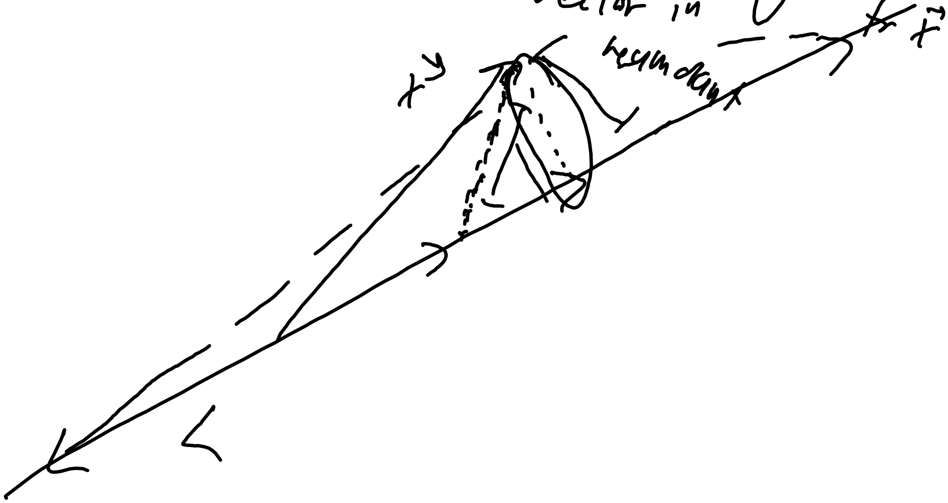
$x+y=1$
$x-y=0$ $\rightsquigarrow$ $x=y=\frac{1}{2}$



orthogonal projection

$\mathrm{proj}_V(\vec{x})$

closest vector in V



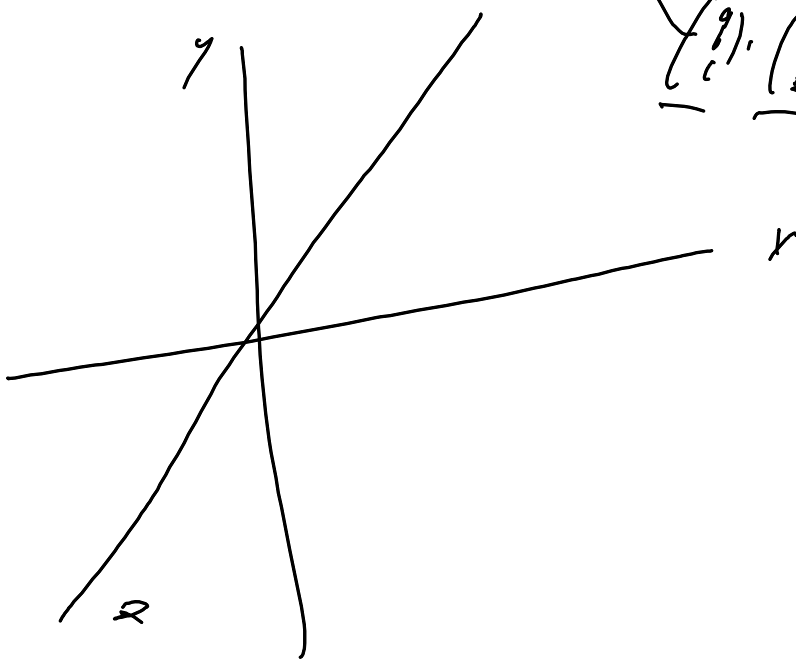
$x+y=1$

$\{\underline{x+y+z=1}\} = P$

$\underline{ax+by+cz=d}$ plane

$\begin{pmatrix}a\\b\\c\end{pmatrix} \leftrightarrow$ normal vect

$\underline{\begin{pmatrix}a\\b\\c\end{pmatrix}} \cdot \underline{\begin{pmatrix}x\\y\\z\end{pmatrix}}$

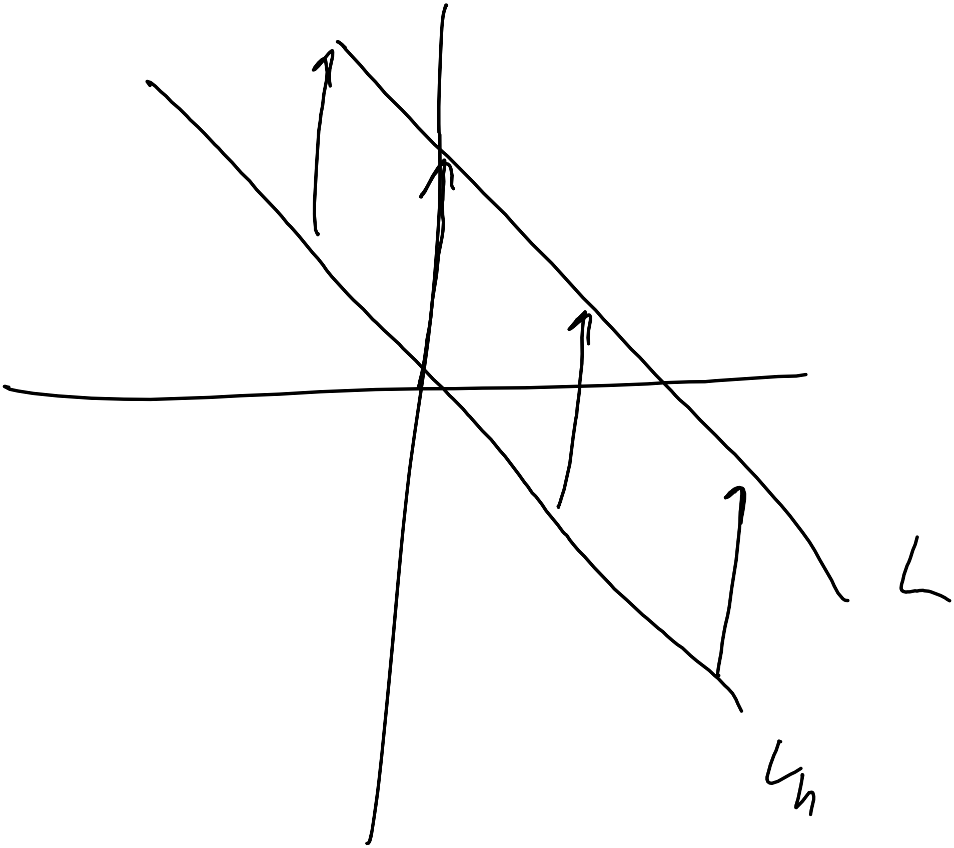


$P_h = \{\underline{x+y+z=0}\}$ subspace of $\mathbb{R}^3$

$\underline{P} = \underline{P_h} + \underline{\begin{pmatrix}1\\0\\0\end{pmatrix}}$

$\vec{v}$ in P

can $\vec{v}_h$ [illegible] plus $\begin{pmatrix}1\\0\\0\end{pmatrix}$



$P_h = \{ x+y+z=0 \}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$ L

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto x+y+z$

$\mathbb{R}^3 \longrightarrow \mathbb{R}$

$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

$$c\vec{v} = \begin{pmatrix} cv_1 \\ \vdots \\ cv_n \end{pmatrix}$$

$$(\vec{v} + \vec{v}') \cdot \vec{w}$$

$$\begin{pmatrix} v_1 + v_1' \\ \vdots \\ v_n + v_n' \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

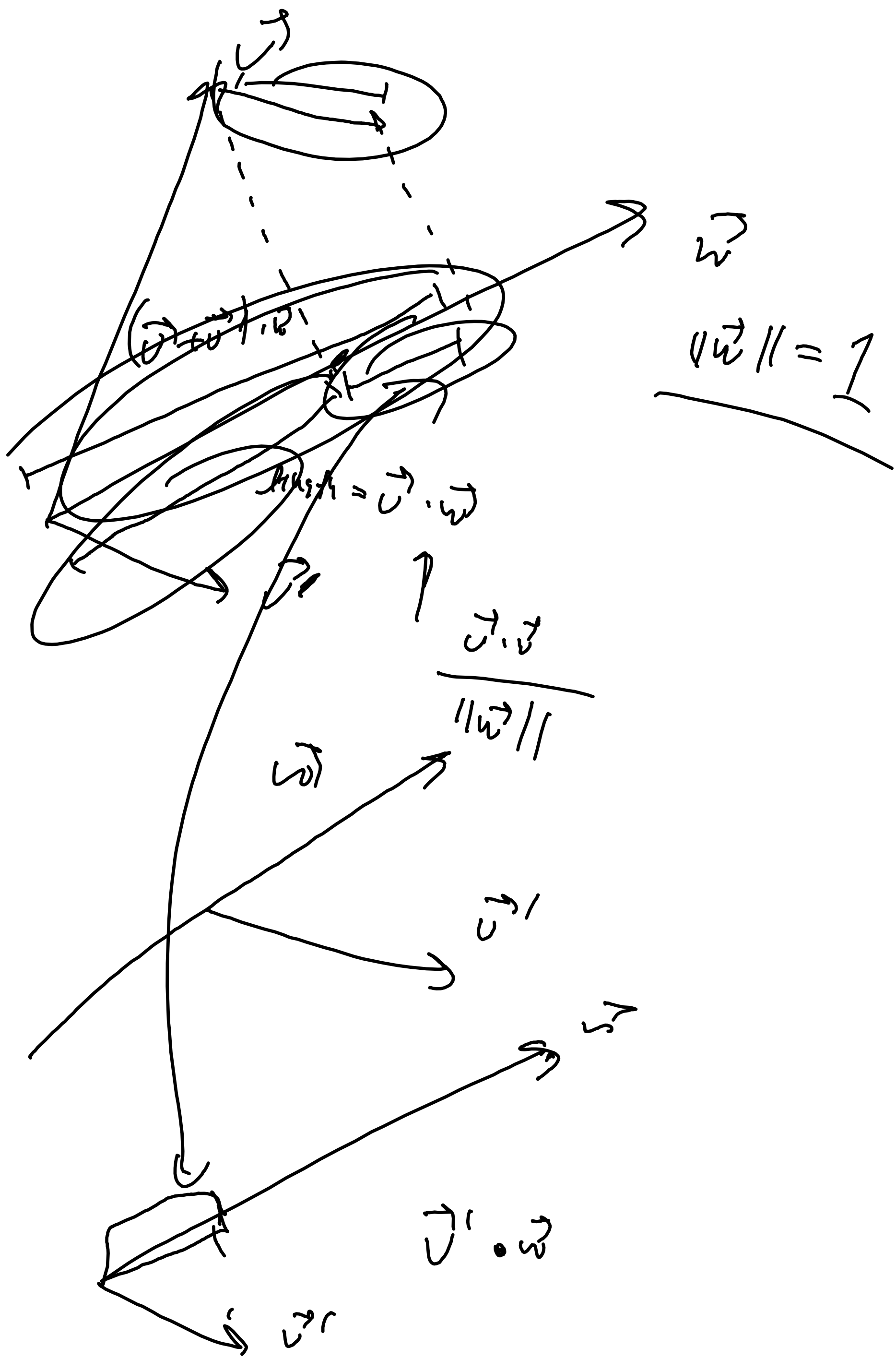
$$= (v_1 + v_1') w_1 + \cdots + (v_n + v_n') \cdot w_n$$

$$= v_1 w_1 + v_1' w_1 + \cdots + v_n w_n + v_n' w_n$$

$$= (v_1 w_1 + \cdots + v_n w_n) + (v_1' w_1 + \cdots + v_n' w_n)$$

$$= \vec{v} \cdot w + \vec{v}' \cdot \vec{w}$$

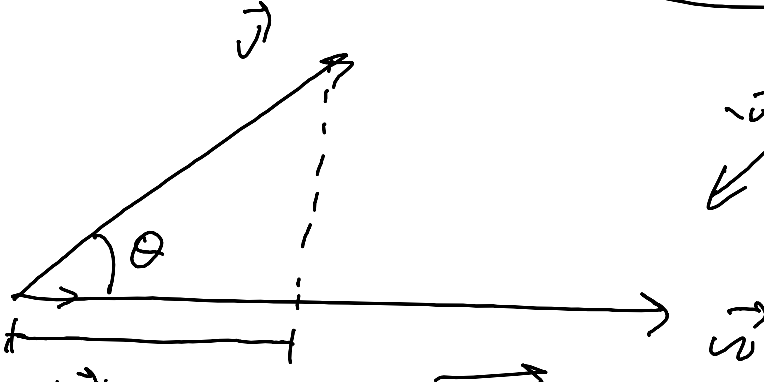
$$\boxed{(\vec{v} + \vec{v}') \cdot \vec{w} = \vec{v} \cdot \vec{w} + \vec{v}' \cdot \vec{w}}$$

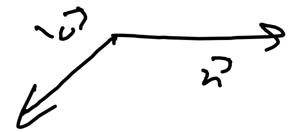


$\vec{v}$
$\vec{w}$
$\|\vec{w}\| = 1$
$(\vec{v}\cdot\vec{w})\vec{w}$
length $= \vec{v}\cdot\vec{w}$
$\vec{v}'$
$\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|}$
$\vec{w}$
$\vec{v}'$
$\vec{w}$
$\vec{v}'\cdot\vec{w}$
$\vec{v}'$

$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$

θ ang between $\vec{v}$, $\vec{w}$





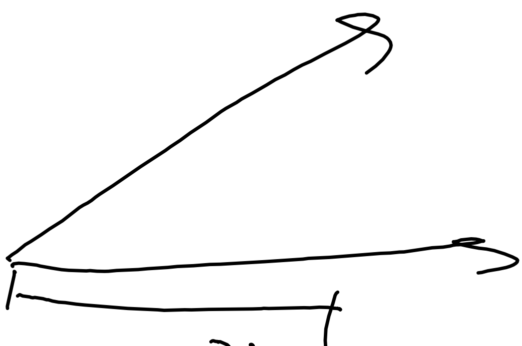
$\|\vec{v}\|\cos\theta$

$\|\vec{v}\|\cos\theta < 0$

$$\|\vec{v}\|\cos\theta = \frac{\|\vec{v}\|\,\|\vec{w}\|\cos\theta}{\|\vec{w}\|} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|}$$

$$\left[\begin{array}{l}(c\vec{v})\cdot\vec{w} = c(\vec{v}\cdot\vec{w}) \\ \vec{v}\cdot(c\vec{w}) = c(\vec{v}\cdot\vec{w})\end{array}\right]$$

$$\underbrace{\left(\|\vec{v}\|\cos\theta\right)}_{\text{proj of } \vec{v} \text{ onto } \vec{w}}\|\vec{w}\|$$

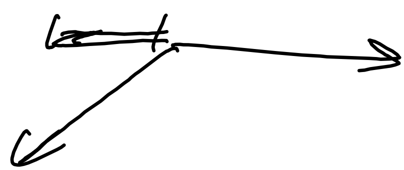


>0

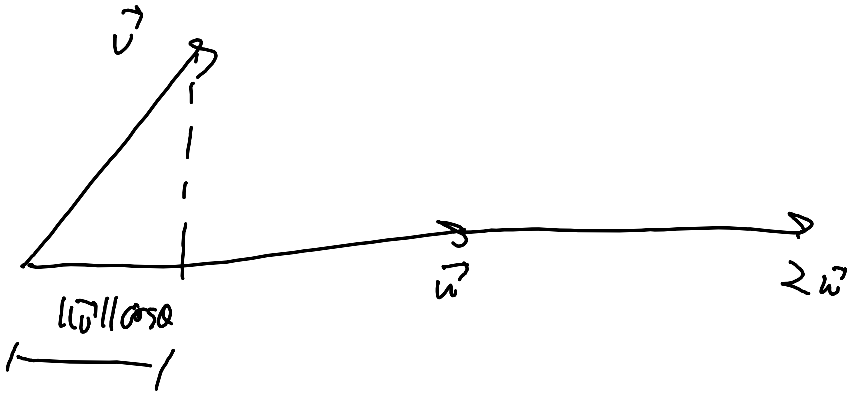
$\|w\|\cos\theta$

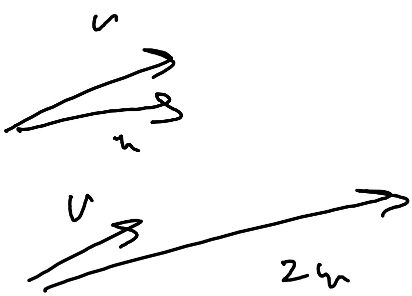
$\|v\|\cos\theta$

<0



$\text{proj}\quad \vec{v} \quad \text{onto} \quad \underbrace{\vec{w}}_{\text{line spanned by } \vec{w}}$





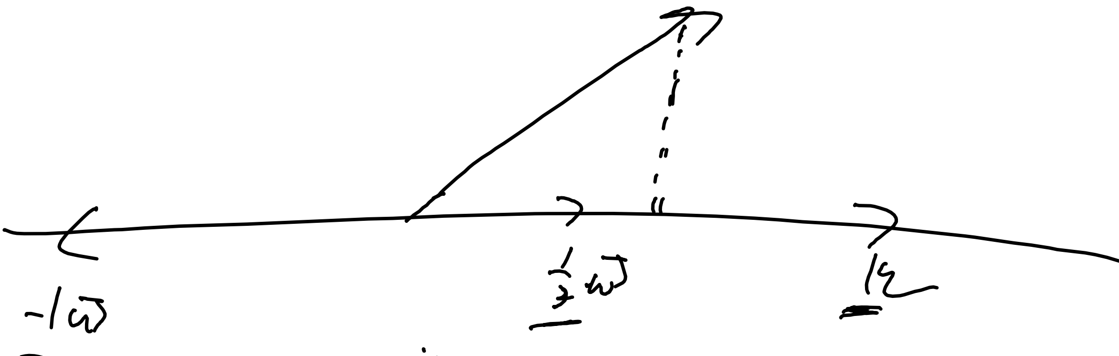
$r = \cos\theta$

$\text{proj}_{\vec{w}}(\vec{v})$

$L = \text{span}(\vec{w})$

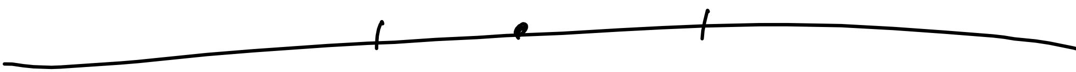
$c\vec{w}$, $c \neq 0$ also spans L

$\text{proj}_L(\vec{v})$



if $\|\vec{w}\| = 1$

$\|\vec{v}\| \cos\theta = \|\vec{v}\| \|\vec{w}\| \cos\theta$



$$\|\vec{v}\|\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|}$$

proj "length"

Only depends on

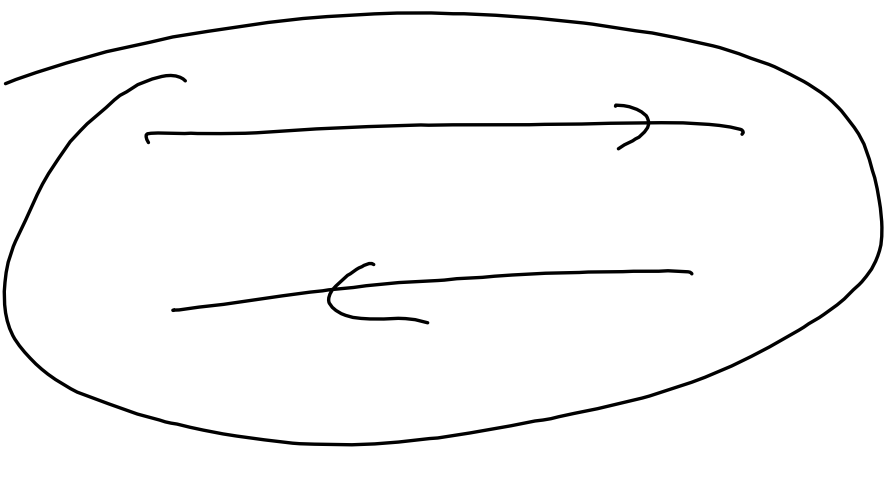
unit on $\vec{w}$

$$\frac{(c\vec{v})\cdot\vec{w}}{\|\vec{w}\|} = c\,\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|}$$

$$\frac{\vec{v}\cdot(c\vec{w})}{\|c\vec{w}\|} = \frac{c\,\vec{v}\cdot\vec{w}}{|c|\,\|\vec{w}\|}$$

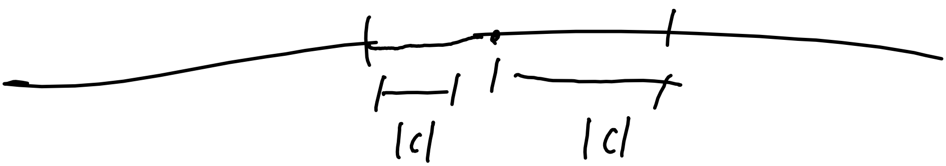
$c>0$

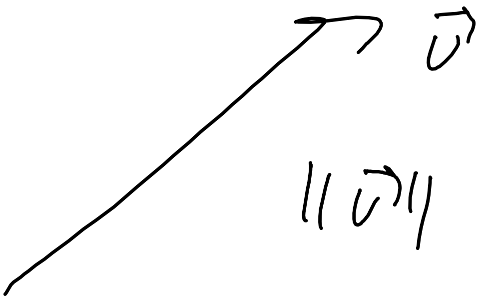
$$= \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|}$$



Scalar c, $|c|$

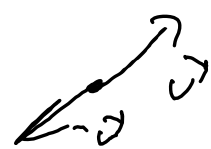
vector $\vec{v}$, $\|\vec{v}\|$





$\|\vec{v}\|$

$$\|c\vec{v}\| = \overbrace{|c|}^{\text{Scala}}\,\overbrace{\|\vec{v}\|}^{\text{vector}}$$



$$= c\|\vec{v}\|$$

$c = -1$

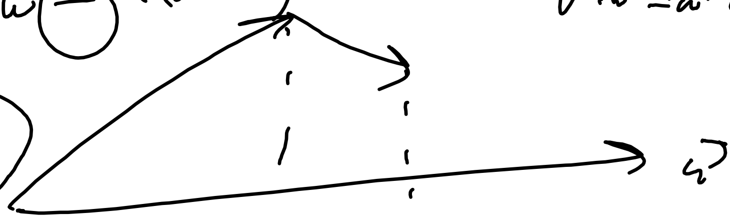
$$\|-\vec{v}\| = -\|\vec{v}\|$$

$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$

$(\vec{v} + \vec{v}') \cdot \vec{w} = \vec{v} \cdot \vec{w} + \vec{v}' \cdot \vec{w}$

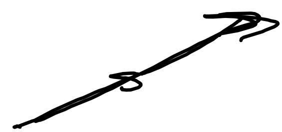
$(c\vec{v}) \cdot w = c(\vec{v} \cdot \vec{w})$ $\qquad$ $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

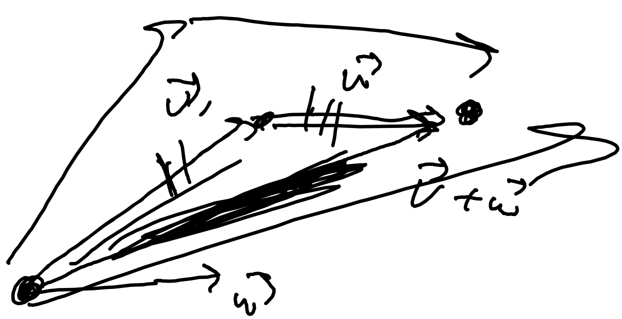
$x^2 \leq x^2 + y^2$



$$\underline{\|\vec{v} + \vec{w}\|} \leq \|\vec{v}\| + \|\vec{w}\|$$

$$(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w})$$





equality $\iff$ $\vec{v}$ $\vec{w}$ parallel

$$\vec{v} \cdot \vec{w} = \sum v_i w_i$$

$$\vec{v} \cdot \vec{v} = v_1^2 + \dots + v_n^2$$

$$= \|\vec{v}\|^2$$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$$

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$$

$$\underline{\vec{v}} \times \underline{\vec{v}} = \vec{0}$$

$$(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \|\vec{v}+\vec{w}\|^2$$

$$= \vec{v}\cdot(\vec{v}+\vec{w}) + \vec{w}\cdot(\vec{v}+\vec{w})$$

$\vec{v}\cdot\vec{w}$
$= \vec{w}\cdot\vec{v}$

$$= \vec{v}\cdot\vec{v} + \underbrace{\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{v}} + \vec{w}\cdot\vec{w}$$

$$= \vec{v}\cdot\vec{v} + 2\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$$

$$= \|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y+z)^2$

$(v+w+u)\cdot(v+w+u)$

$$\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}^2\|$$

$$||\vec{v}+\vec{w}|| \underset{(?)}{\le} ||\vec{v}|| + ||\vec{w}||$$

goal

$a, b \geq 0$

then

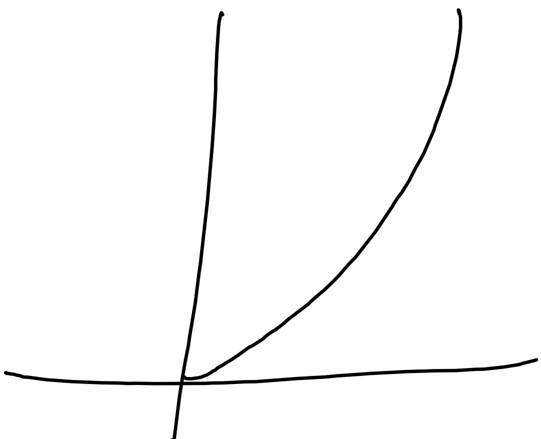
$a^2 \le b^2$

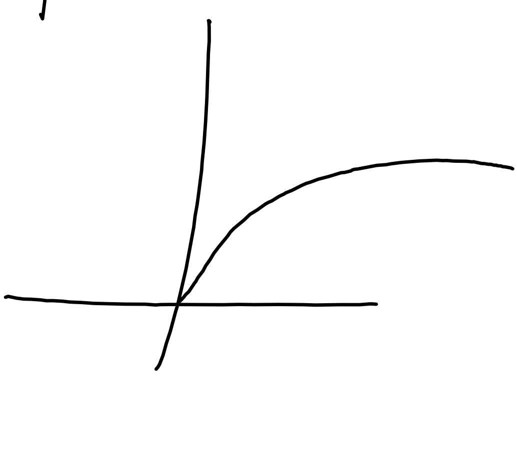
iff

$a \le b$

square $\Downarrow$ $\Updownarrow$ $\Uparrow$ square root

$$||\vec{v}+\vec{w}||^2 \underset{(?)}{\le} \left(||\vec{v}|| + ||\vec{w}||\right)^2$$





$$\|\vec{v}+\vec{w}\|^2 \overset{(?)}{\leq} (\|\vec{v}\| + \|\vec{w}\|)^2$$

$$\Updownarrow \qquad\qquad \Downarrow$$

$$\|\vec{v}\|^2 + 2\cancel{\vec{v}\cdot\vec{w}} + \|\vec{w}\|^2 \overset{(?)}{\leq} \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2$$

$(\vec{v}\cdot\vec{w}=0)$

$$\Updownarrow$$

$$2\vec{v}\cdot\vec{w} \overset{(?)}{\leq} 2\|\vec{v}\|\|\vec{w}\|$$

$$\vec{v}\cdot\vec{w} \overset{(?)}{\leq} \|\vec{v}\|\|\vec{w}\| \checkmark$$

$(?) \Leftarrow \checkmark$

$$\sqrt{(\vec{v}\cdot\vec{w})^2} = |\vec{v}\cdot\vec{w}|$$

$$\Downarrow$$

$$\sqrt{\vec{v}^2\cdot\vec{w}^2}$$

$\vec{v}^2$

$\underset{\text{vect}}{—} \cdot \underset{\text{vect}}{—}$

$\vec{v}^2$

$$(\vec{v}\cdot\vec{w})^2$$

$$\|$$

$$\cancel{\vec{v}^2\cdot\vec{w}^2}$$

$$\|$$

$$\|\vec{v}\|^2\|\vec{w}\|^2$$

$$(xy)^2 = x^2y^2$$

$$(\vec{v}\cdot\vec{w})^2 \leq \|\vec{v}\|^2\|\vec{w}\|^2$$

Cauchy-Schwarz

$$\vec{v}\cdot\vec{w} \leq \|\vec{v}\|\,\|\vec{w}\|$$

(!)

mult by 2 // add $\|v\|^2+\|w\|^2$

$$\|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|w\|^2 \leq \|\vec{v}\|^2 + 2\|\vec{v}\|\,\|\vec{w}\| + \|\vec{w}\|^2$$

(!)

// simplify both sides

$$\|\vec{v}+\vec{w}\|^2 \leq (\|\vec{v}\|+\|\vec{w}\|)^2$$

(!)

$\Downarrow \sqrt{}$ both sides

$$\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$$

$\Delta \le$

$$\boxed{\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta}$$

$$\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \cos\theta$$

$\perp$

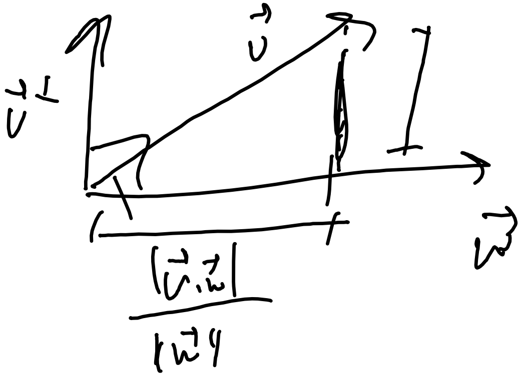
what if this is S?

$$|\cos\theta| \le 1$$

here

$$\left|\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|\,\|\vec{w}\|}\right| \le 1 \iff |\vec{v}\cdot\vec{w}| \le \|\vec{w}\|\,\|\vec{w}\|$$

C-S



$$|\vec v\cdot\vec w| \le \|\vec v\|\|\vec w\|$$

$$\vec v^\perp = \vec v - \underbrace{\frac{(\vec v\cdot\vec w)}{\vec w\cdot\vec w}}\vec w \qquad \vec w \ne 0$$

$$\underline{\vec v^\perp\cdot\vec w} = 0$$

$$\left(\vec v - \frac{\vec v\cdot\vec w}{\vec w\cdot\vec w}\vec w\right)\cdot\vec w$$

$$= \vec v\cdot\vec w - \frac{\vec v\cdot\vec w}{\cancel{\vec w\cdot\vec w}}\cancel{\vec w\cdot\vec w}$$

Scala

$$=0$$

$$\|\vec v\|^2 = \|\vec v^\perp\|^2 + \left\|\overbrace{\frac{\vec v\cdot\vec w}{\vec w\cdot\vec w}}\vec w\right\|^2$$

$$\|\vec v\|^2 - \|\vec v^\perp\|^2 = \left|\frac{\vec v\cdot\vec w}{\cancel{\vec w\cdot\vec w}}\right| \cancel{\|w\|^2}$$

$$\|\vec v\|^2 - \|v^\perp\|^2 = |\vec v\cdot\vec w|$$

$5, 1, 3, 3$

$n = 2.$

$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

$x + y = 1$

$x + y = 1$

$?$

$x^2 + y^2$

$1 = (x+y)^2$

$x+y=1$

$1 = x^2 + 2xy + y^2$

$1 - 2xy = x^2 + y^2$

min $1 - 2xy$

s.t. $x + y = 1$

$x = 1/2$

$y = 1/2$

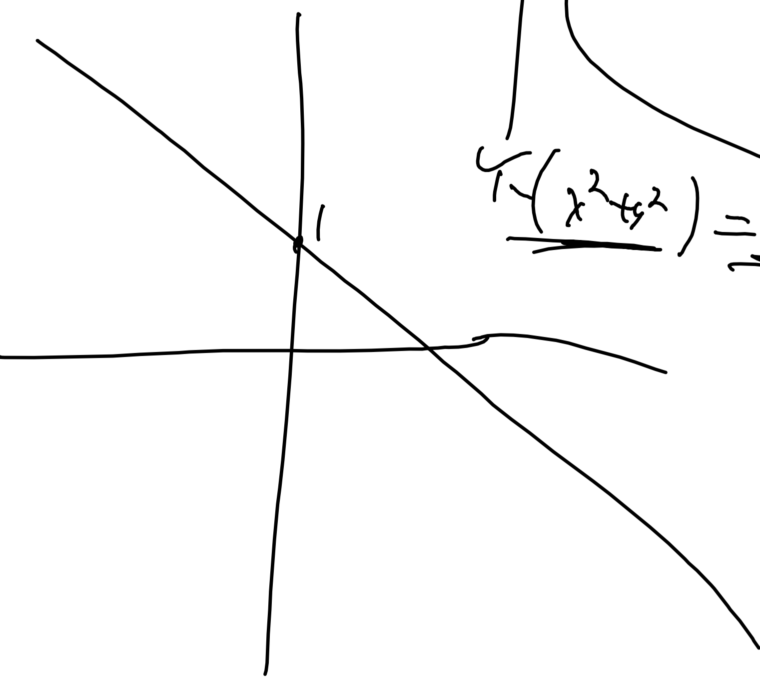
$\boxed{g = 0}$

$x^2 + xy$

$\sin(x) - \log(y) = 1$

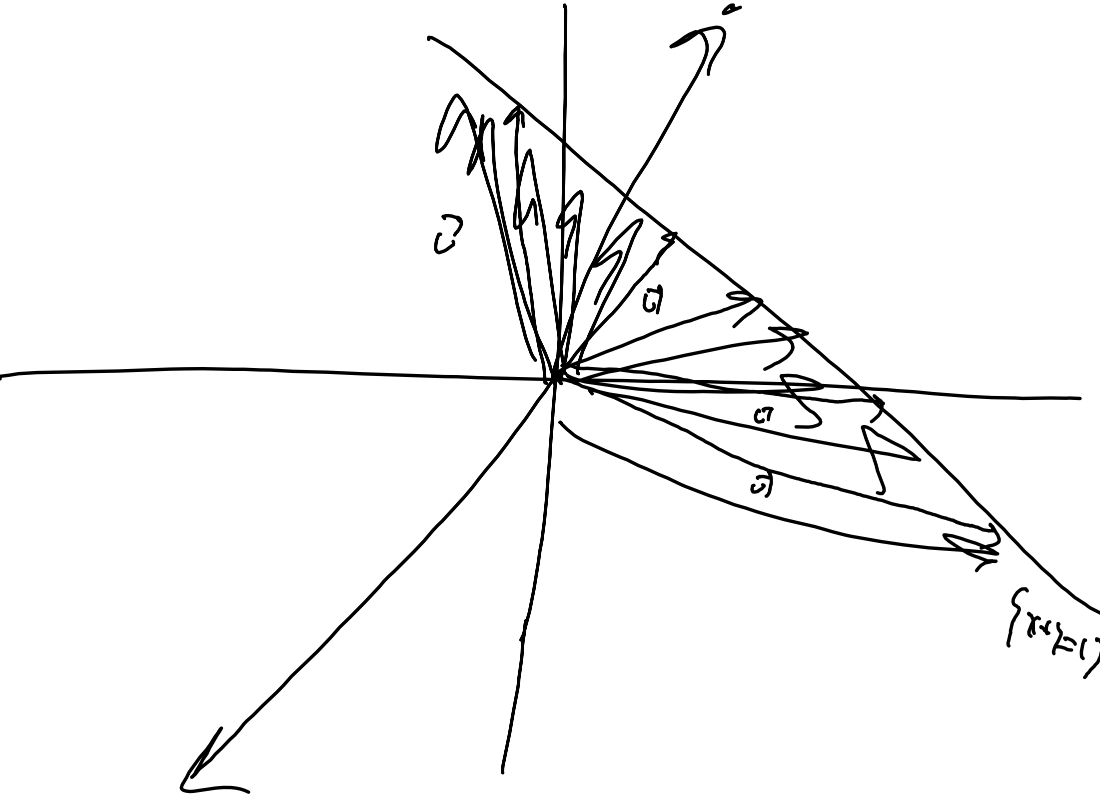
$g = 0$

$1 - (x^2 + y^2) = 2xy$



$m = -1$

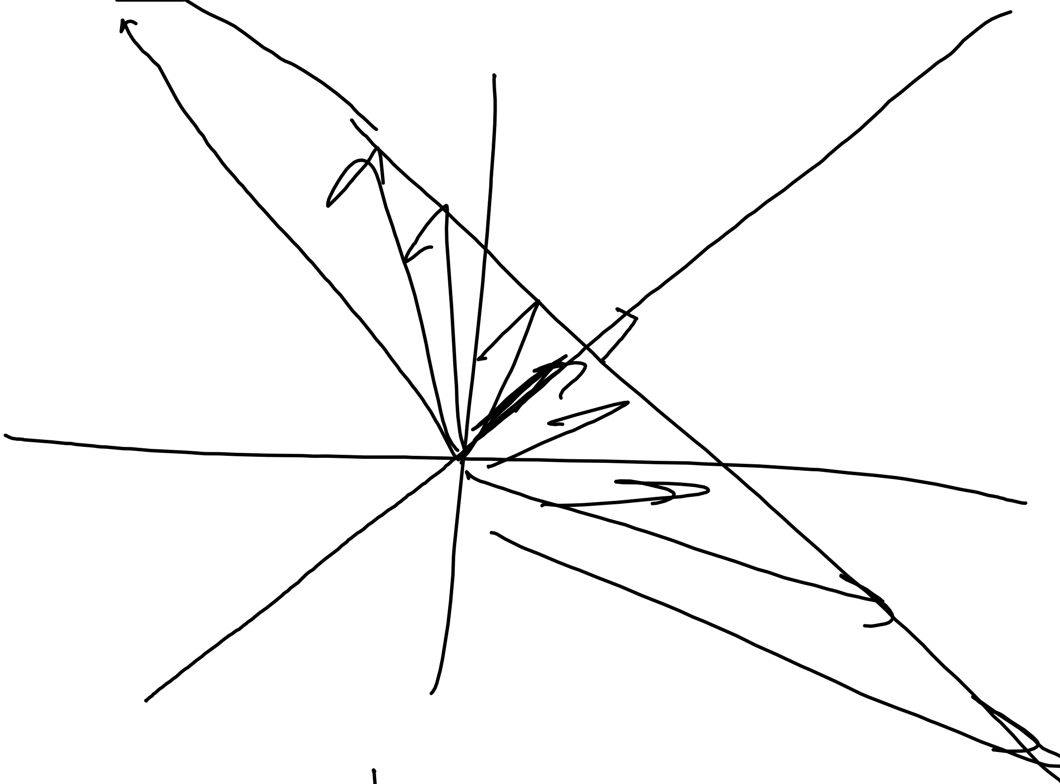
$\{x + y = 1\}$

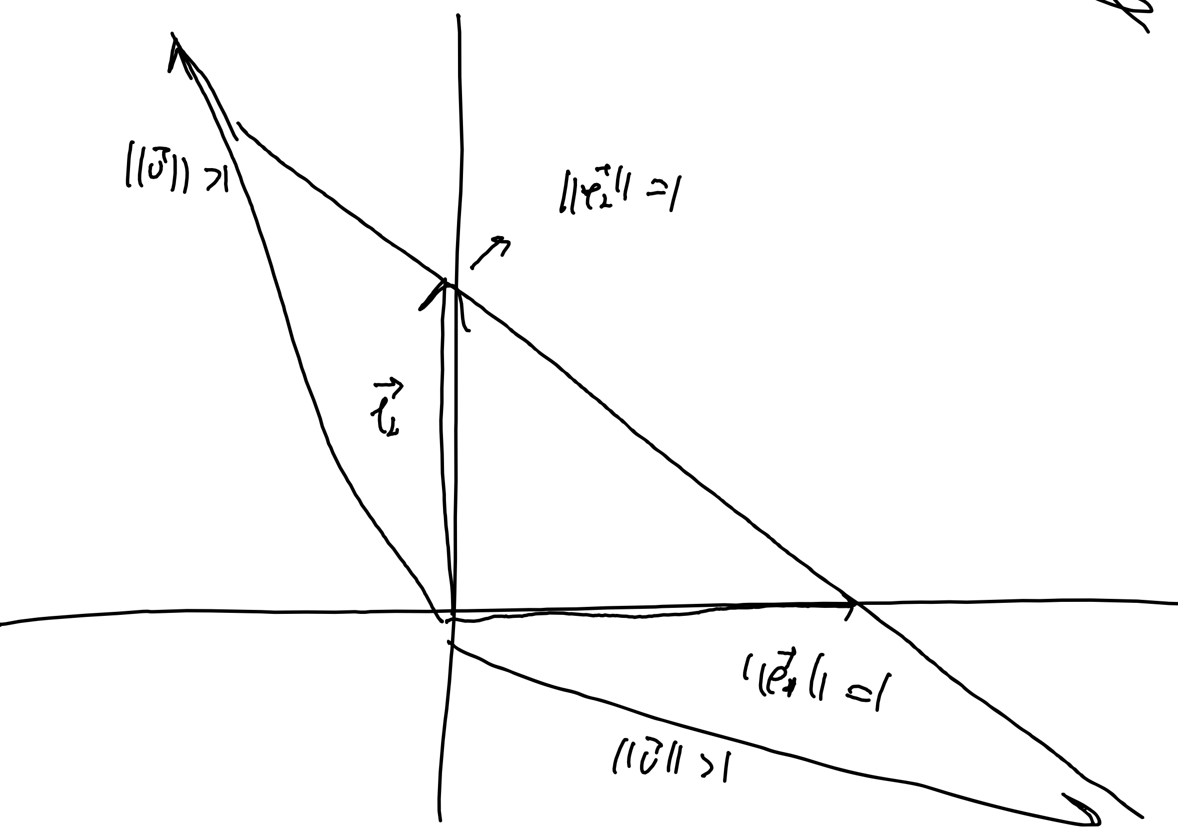


$$\cos^2\theta + \sin^2\theta = 1$$

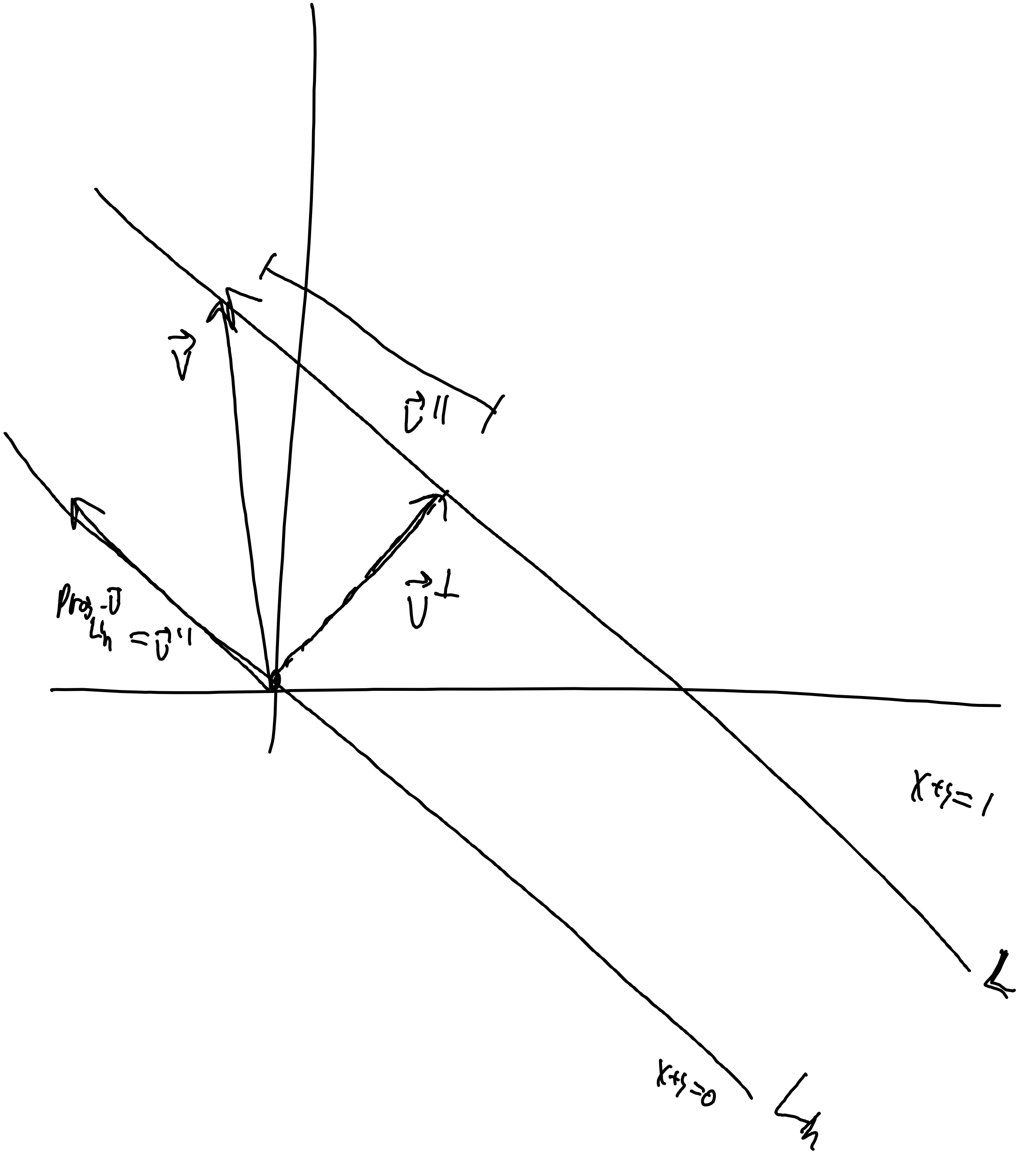
$$x^2 + y^2$$

$$\underline{x+y}$$

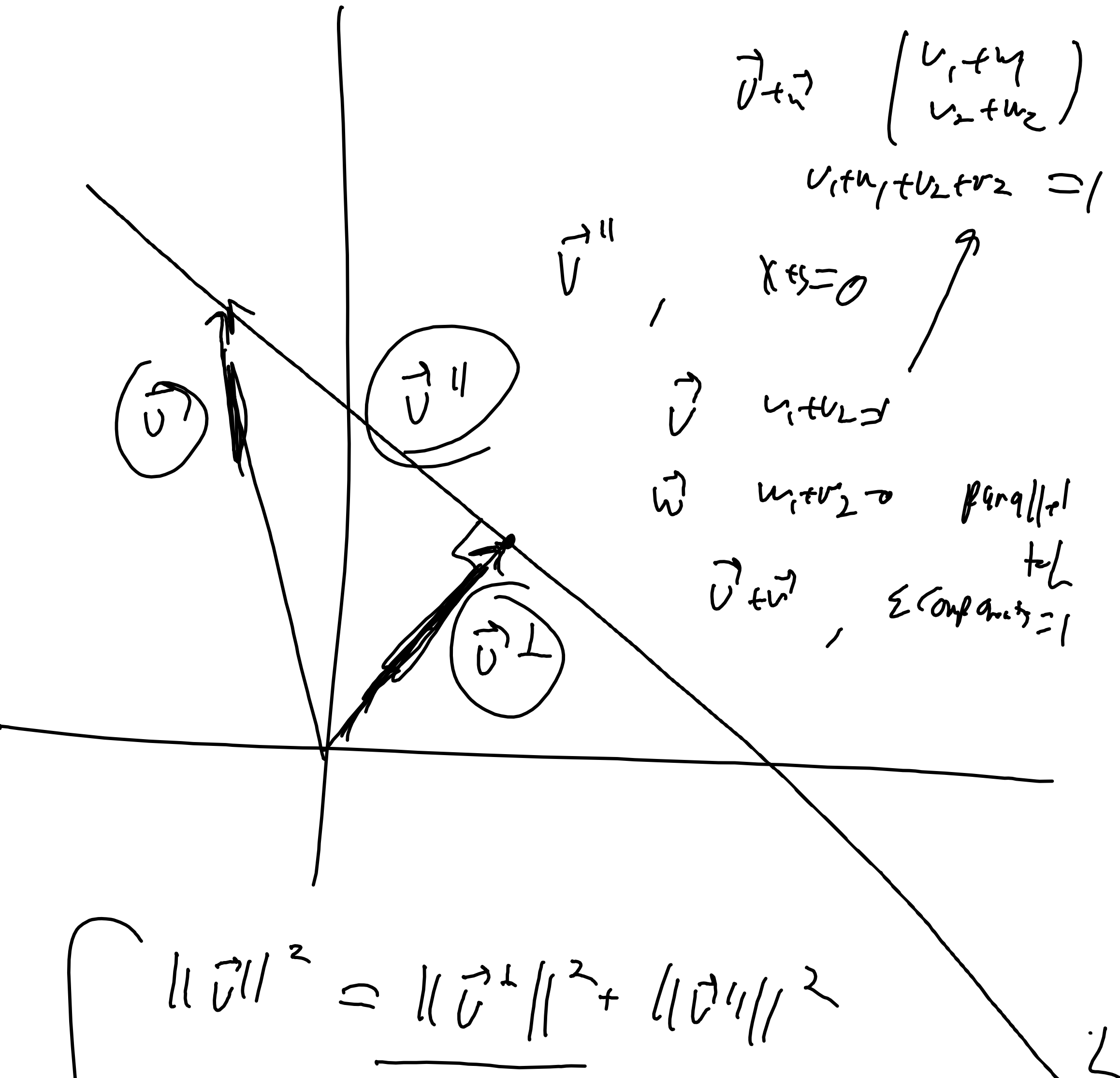




$\|\vec{v}\| > 1$
$\|\vec{e_2}\| = 1$
$\vec{e_2}$
$\|\vec{e_1}\| = 1$
$\|\vec{v}\| > 1$



$\vec{v}$
$\vec{v}^{\parallel}$
$\vec{v}^{\perp}$
$\text{Proj}_{L_h} \vec{v} = \vec{v}^{\parallel}$
$x+y=1$
L
$x+y=0$
L_h



$\vec{v}+\vec{w}$ $\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \end{pmatrix}$

$v_1+w_1+v_2+w_2=1$

$\vec{v}^{\parallel}$, $x+y=0$

$\vec{v}$ $v_1+v_2=1$

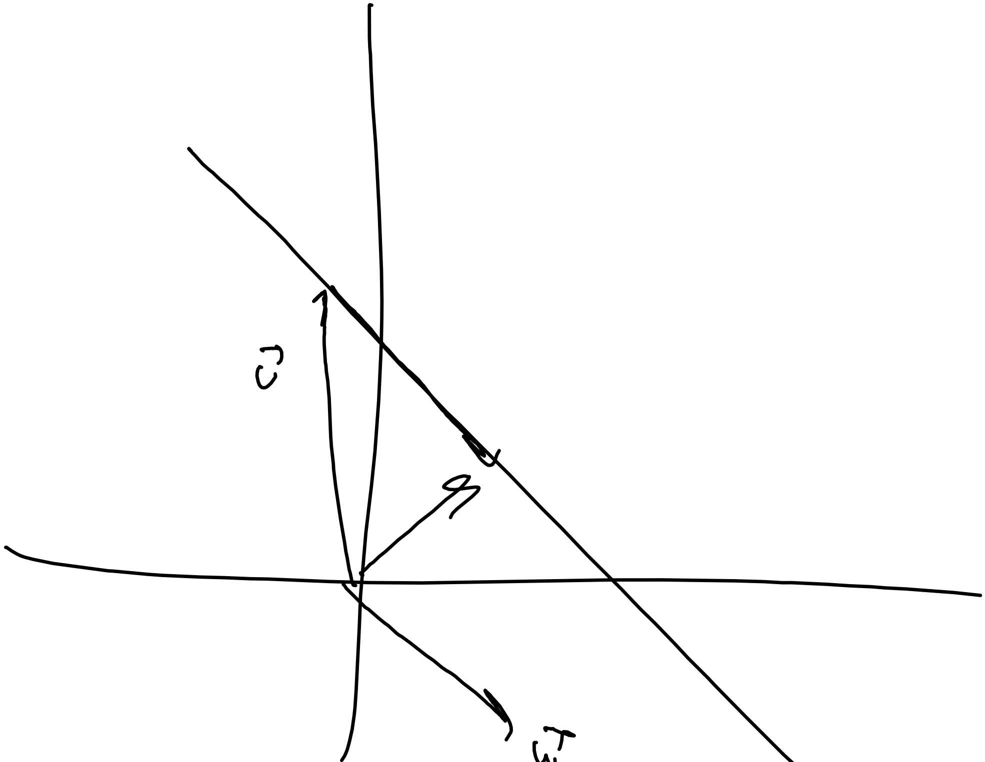
$\vec{w}$ $w_1+w_2=0$ parallel to L

$\vec{v}+\vec{w}$, Σ components $=1$

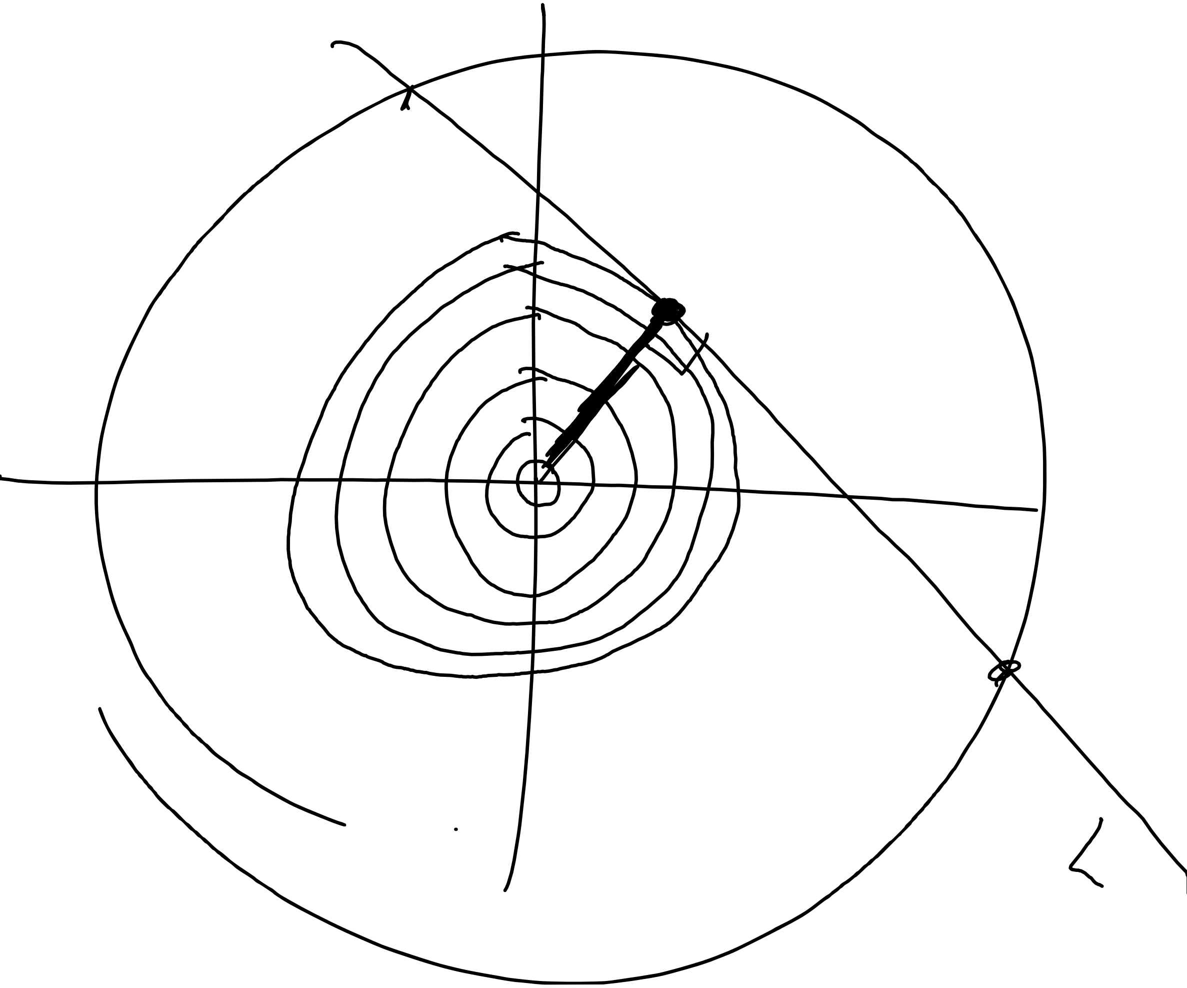
$$\|\vec{v}\|^2 = \|\vec{v}^{\perp}\|^2 + \|\vec{v}^{\parallel}\|^2$$

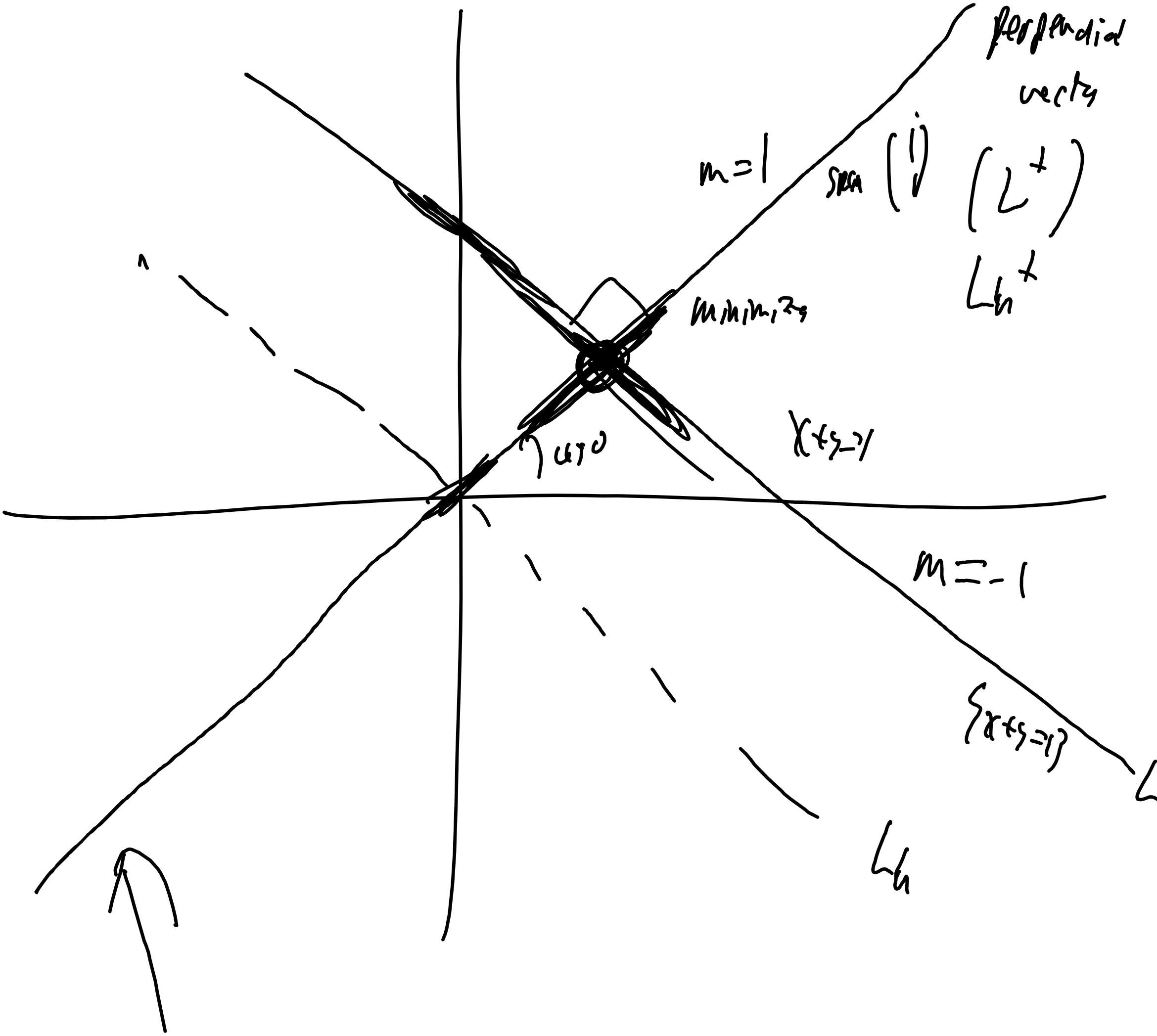
$$\|\vec{v}^{\perp}\|^2 = \|\vec{v}\|^2 - \|\vec{v}^{\parallel}\|^2 \le \|\vec{v}\|^2$$

$\vec{v}^{\parallel}=0$ is equality

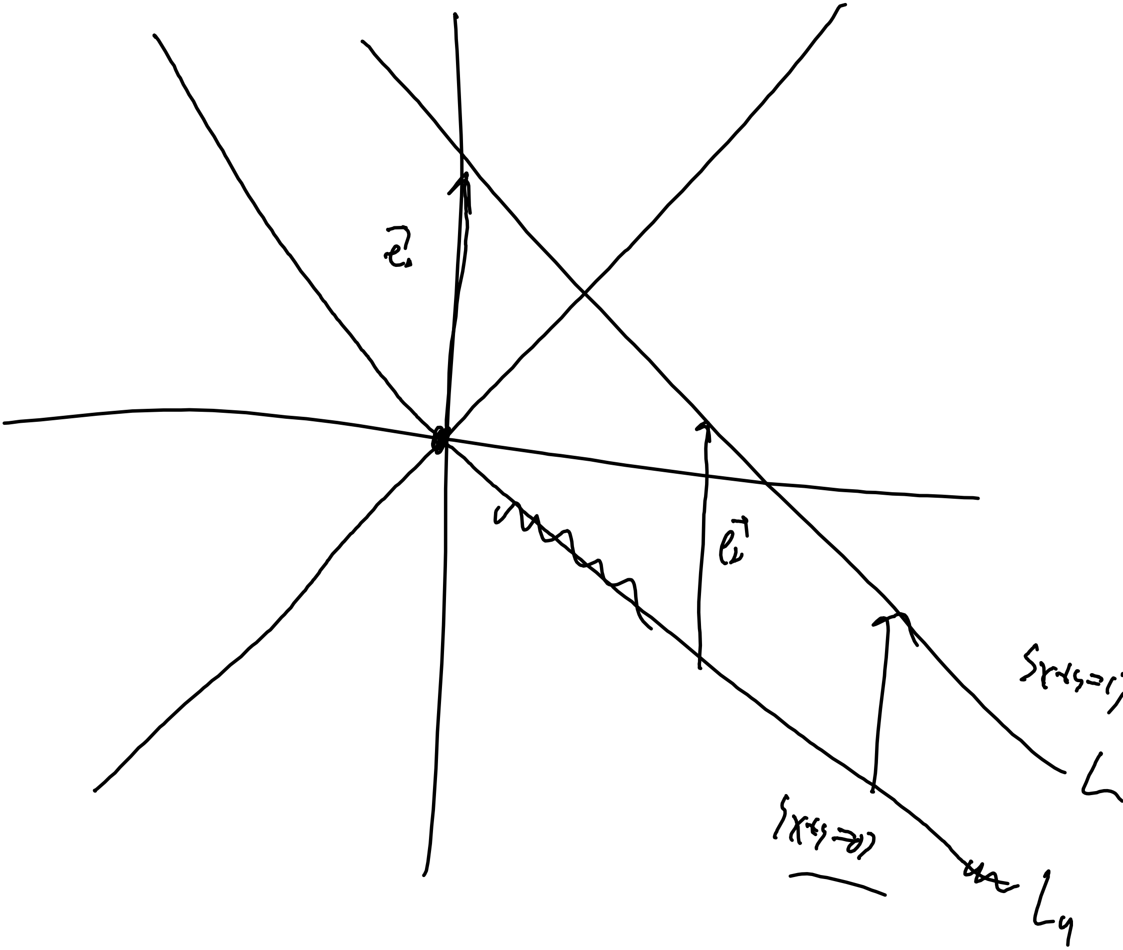


$\text{proj}_V(\vec{x}) = \text{closest } \vec{v} \in V \text{ to } \vec{x}$





how do we find this line?



$\{x+y=0\}$

$$x+y = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = 0$$

$\Leftrightarrow$

$$\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \in L_u \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\operatorname{span}\begin{pmatrix}1\\1\end{pmatrix} \perp L_y$

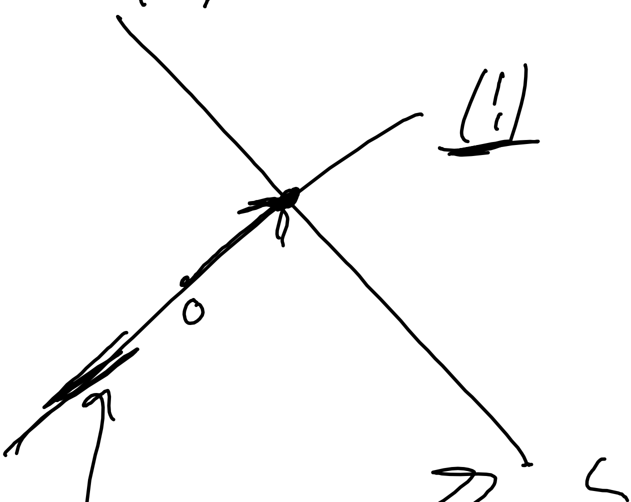
$= \{x+y=0\}$

$\begin{pmatrix}x\\y\end{pmatrix} \longmapsto x+y$

$\mathbb{R}^2 \longrightarrow \mathbb{R}$

$\underline{\begin{pmatrix}1 & 1\end{pmatrix}}$

~~$x+y=1$~~

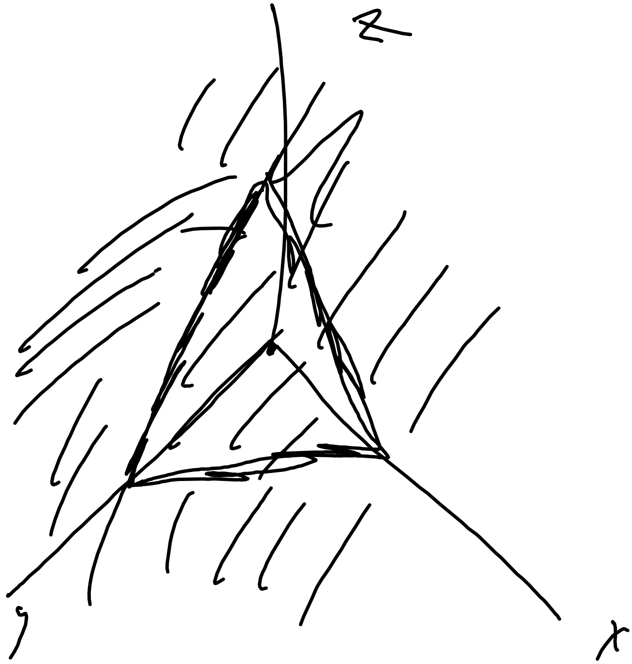


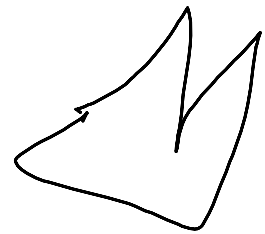
$x+y=1$

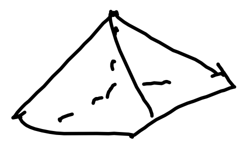
$\underline{x=y}$

$\boxed{x=1/2 \quad y=1/2 \qquad \begin{pmatrix}1/2\\1/2\end{pmatrix}}$

$x+y+z=1$







$\{x+y+z=0\} = \ker$, 2d

$\begin{pmatrix}x\\y\\z\end{pmatrix} \longmapsto x+y+z$

$\mathbb{R}^3 \longrightarrow \mathbb{R}$

$\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$, Orthog. complement

Span $\begin{pmatrix}1\\1\\1\end{pmatrix}$

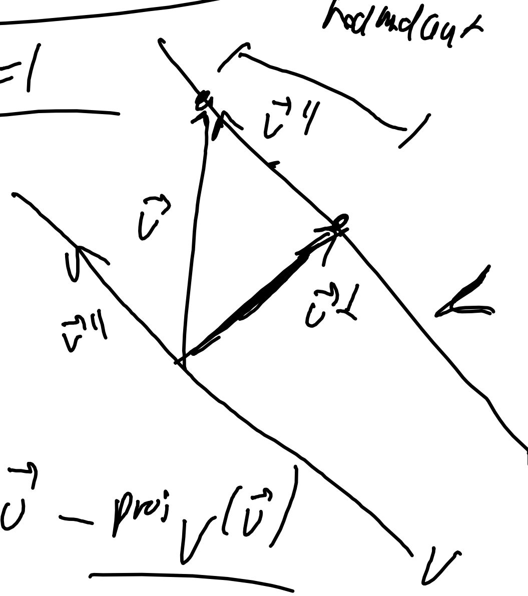
$\begin{pmatrix}x\\y\\z\end{pmatrix} \in \text{Span}\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad x=y=z$

$x+y+z=1 \longrightarrow \begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix}$

$\vec{v}$ $\underline{v_1 + v_2 + \dots + v_n = 1}$

$V = \{ v_1 + v_2 + \dots + v_n = 0 \}$

Let $\vec{v}$ have $\underline{v_1 + v_2 + \dots + v_n = 1}$



$\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$

in V in $V^{\perp}$

$\vec{v}^{\parallel} = \mathrm{proj}_V(\vec{v})$

$\vec{v}^{\perp} = \vec{v} - \underline{\mathrm{proj}_V(\vec{v})}$

$\|\vec{v}\|^2 = \|\vec{v}^{\parallel}\|^2 + \|\vec{v}\|^{\perp 2}$

$\vec{v}^{\parallel} \perp \vec{v}^{\perp}$

$\|\vec{u}\| \leq \|\vec{v}^{\perp}\| \leq \underline{\|\vec{v}\|}$

$\vec{v}^{\perp}$ also has sum components $= \underline{1}$

$$\vec{v} = \underline{\vec{v}^{\parallel}} + \underline{\vec{v}^{\perp}}$$

$$\mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\vec{v} \longmapsto v_1 + \dots + v_n$$

$$v_1 + \dots + v_n = 1$$

$\vec{v}^{\parallel}$ in V

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \vec{v}^{\parallel} = \begin{pmatrix} v_1^{\parallel} \\ \vdots \\ v_n^{\parallel} \end{pmatrix} \to \underline{v_1^{\parallel} + \dots + v_n^{\parallel} = 0}$$

$$\vec{v}^{\perp} = \begin{pmatrix} v_1^{\perp} \\ \vdots \\ v_n^{\perp} \end{pmatrix}$$

$$v_i = v_i^{\parallel} + v_i^{\perp}$$

$$1 = v_1 + \dots + v_n$$

$$= (v_1^{\parallel} + v_1^{\perp}) + (v_2^{\parallel} + v_2^{\perp}) + \dots + (v_n^{\parallel} + v_n^{\perp})$$

$$= \underbrace{(v_1^{\parallel} + \dots + v_n^{\parallel})}_{=0} + \underline{(v_1^{\perp} + \dots + v_n^{\perp})}$$

$$= v_1^{\perp} + \dots + v_n^{\perp}$$

$\vec{v}^{\perp}$ is a valid vector to consider

sum components of $\vec{v}^{\perp}$ is 1 from $\vec{v}$ s.t. $\sum v_i = 1$

form $\vec{v}^\perp$ s.t. $\sum v_i^\perp = 1$

Claim $\|\vec{v}^\perp\| \leq \|\vec{v}\|$

if $\vec{v}$ not perp to V

$\underline{\|\vec{v}^\perp\|}^2 < \underline{\|\vec{v}\|}^2$

differ $\|\mathrm{proj}_V(\vec{v})\|^2$

Proven: If we minimize length s.t. sum components $= 1$

They must be perp to V

$V^\perp$, $V = \{v_1 + \dots + v_n = 0\}$

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \vec{v} = 0$

$\|$

span $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\vec{v}$

$u_1 + \dots + u_n = 1$

$\vec{v} \in \operatorname{span}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \Leftarrow u_1 = u_2 = \dots = u_n$

$u_i = 1/n$

$$\begin{pmatrix}1/n\\1/n\\\vdots\\1/n\end{pmatrix}$$