$\boxed{\text{wksht 5}}$

problem 1

next: challenges

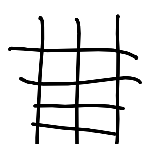
$\vec{v}, \vec{w}$ vectors in $\mathbb{R}^n$

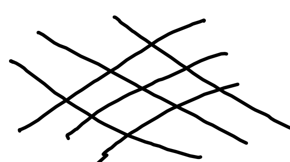
$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

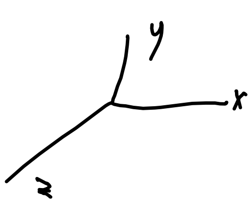
$(v_1 \cdots v_n) \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

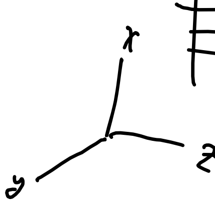
$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

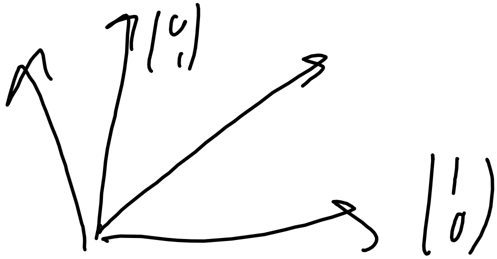
Concerns:
- $2v_1w_1 + 3v_2w_2 + \dots + 17v_nw_n$ arbitrary?
- coordinate dependent









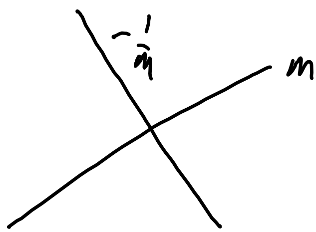


role of basis?

orthonormal

geometry?

$\vec{v} \cdot \vec{w} = 0 \iff \vec{v}, \vec{w}$ perp?



$$\boxed{\vec{v} \cdot \vec{w} \overset{(?)}{=} \|\vec{v}\| \, \|\vec{w}\| \cos\theta}$$

θ angle between $\vec{v}, \vec{w}$

def of θ

$$(\vec{v}_1 + \vec{v}_2) \cdot \vec{w} = \vec{v}_1 \cdot w + \vec{v}_2 + w$$

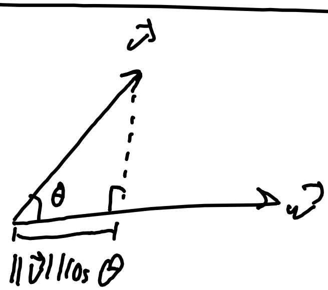
$$\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2$$

$$(c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (cw) = c(\vec{v} \cdot \vec{w})$$

algebraic

$$\left(\sum (c_i \vec{v}_i) \cdot \vec{w} = \sum c_i (\vec{v}_i \cdot w)\right)$$

$$\|\vec{v}\| \, \|\vec{w}\| \, \cos(\theta)$$



$\vec{v}$ proj onto $\vec{w}$

take the length of proj

mult. by length of $\|\vec{w}\|$

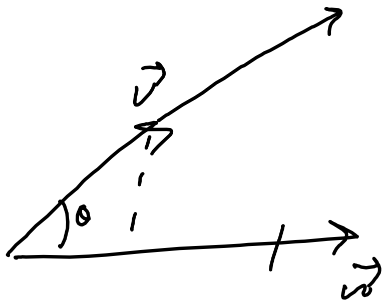
Def. $D(\vec{v}, \vec{w}) = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$

lengths

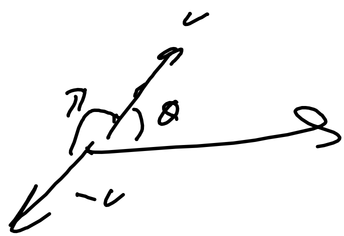
angle

Goal: $\vec{v} \cdot \vec{w} = D(\vec{v}, \vec{w})$

$D(c\vec{v},\vec{w}) = \underline{c}\; D(\vec{v},\vec{w})$



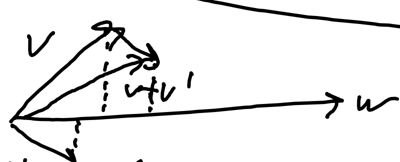
$\left.\begin{array}{ll} \theta & \vec{v},\vec{w} \\ \theta & 2\vec{v},\vec{w} \end{array}\right\}$ same θ



$\cos(\theta+\pi) = -\cos(\theta)$

$D(\vec{v}, c\vec{w}) = cD(\vec{v},\vec{w})$

$D(\vec{v}+\vec{v}',\vec{w}) \overset{?}{=} D(\vec{v},\vec{w}) + D(\vec{v}',\vec{w})$



$\|v+v'\| \, \|w\| \cos(\varphi)$

$\|v\| \, \|w\| \cos\theta + \|v'\| \, \|w\| \cos\theta'$

φ ang $\vec{v}+\vec{v}'$, $\vec{w}$

θ " $\vec{v}$, $\vec{w}$

θ' " $\vec{v}'$, $\vec{w}$

defect D, algebraic?

upshot D, symmetry under "rigid transformations"

R rotation

$D(R\vec{v}, R\vec{w}) = D(\vec{v}, \vec{w})$ only care about length + angle

$\left((R\vec{v}) \cdot (R\vec{w}) = \vec{v} \cdot \vec{w} \, ?? \right)$

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $a^2+b^2=1$

goal: $D(\vec{v}+\vec{v}\,', \vec{w}) = D(\vec{v},\vec{w}) + D(\vec{v}\,',\vec{w})$

$\vec{w}=\vec{0}$, both sides are 0

$D(\vec{v},\vec{w}) = \|\vec{v}\|\,\|\vec{w}\|\cos(\theta)$

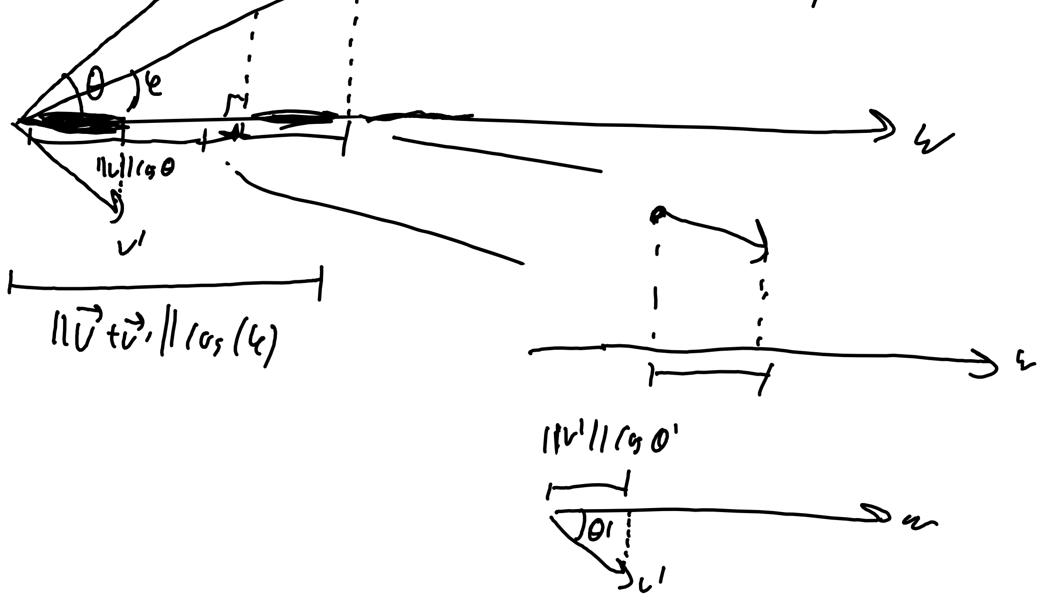
θ angle between $\vec{v},\vec{w}$.

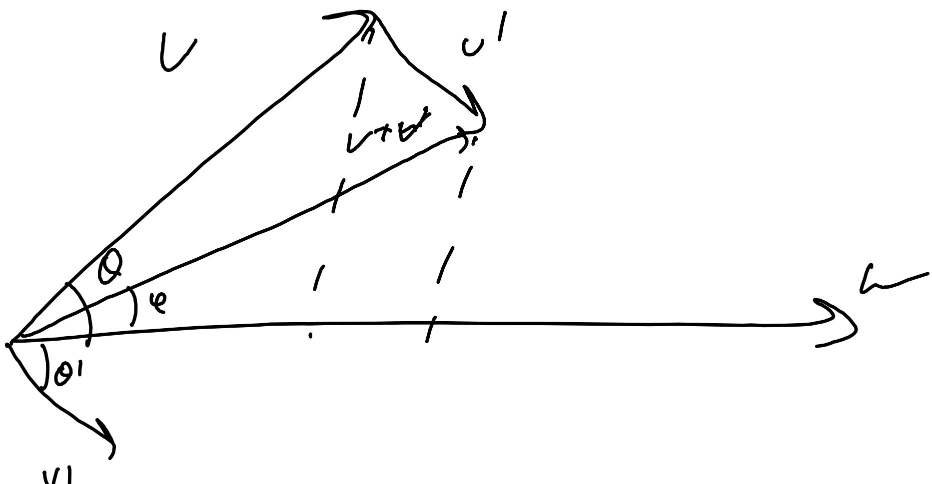
$D(\vec{v}+\vec{v}\,',\vec{w})$

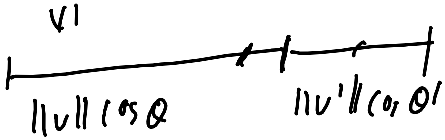
θ ang between $\vec{v},\vec{w}$

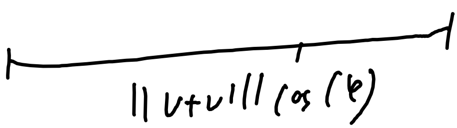
φ ang between $\vec{v}+\vec{v}\,',\vec{w}$

θ' ang between $\vec{v}\,',\vec{w}$









$$\|\vec{v}\|\cos\theta + \|\vec{v}'\|\cos\theta' = \|\vec{v}+\vec{v}'\| \cos(\varphi)$$

$$\shortparallel \qquad \shortparallel \qquad \shortparallel$$

$$\frac{D(\vec{v},\vec{w})}{\|\vec{w}\|} + \frac{D(\vec{v},\vec{w})}{\|\vec{w}\|} = \frac{D(\vec{v}+\vec{v}',\vec{w})}{\|w\|}$$

$\vec{w} \neq 0$

- $D(\vec{v},\vec{w})+D(\vec{v}',\vec{w})=D(\vec{v}+\vec{v}',\vec{w})$ ✓

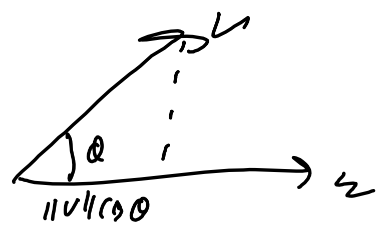
- $D(\vec{v},\vec{w})=D(\vec{w},\vec{v})$

$D(\vec{v},\vec{w}+\vec{w}')=D(\vec{v},\vec{w})+D(\vec{v},\vec{w}')$

$$\vec{v}\cdot\vec{w}=D(\vec{v},\vec{w})=\|\vec{v}\|\|\vec{w}\|\cos(\theta)$$

alg

geom



$\vec{e_1}\cdot\vec{e_1}$

$\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$

$=1\cdot1+0\cdot0+\dots+0\cdot0$

$=1$

$D(\vec{e_1},\vec{e_1})$

$\|\vec{e_1}\|\|\vec{e_1}\|\cos(\theta)$

$\|$

$1\cdot1$

$=1$

$\theta=0$

$\cos\theta=1$

$\|\vec{e_1}\|=1$

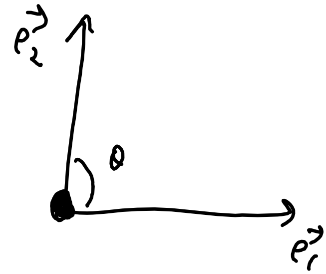
$\vec{e_1}\cdot\vec{e_2}$

$\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\\ \vdots\\0\end{pmatrix}$

$1\cdot0+0\cdot1+0\cdot0+\dots+0\cdot0$

$=0$

$$D(\vec{e_1}, \vec{e_2}) = ?$$



$$\|\vec{e_1}\|\,\|\vec{e_2}\| \cos(\theta)$$

$$\theta = \pi/2$$

$$\cos\theta = 0$$

$$\vec{e_i} \cdot \vec{e_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$D(\vec{e_i}, \vec{e_j}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\vec{v} \cdot \vec{w} = \left(\sum_i v_i \vec{e_i}\right) \cdot \left(\sum_j w_j \vec{e_j}\right)$$

$$= \sum_{ij} v_i w_j \underbrace{\vec{e_i} \cdot \vec{e_j}}$$

$$= \sum_i v_i w_i$$

$$\vec{v} \cdot \vec{w} = D(\vec{v}, \vec{w})$$

$$D\left(\sum v_i \vec{e_i}, \sum w_j \vec{e_j}\right) = \sum_{ij} v_i w_j D(\vec{e_i}, \vec{e_j})$$

$$= \sum_i v_i w_i = \vec{v} \cdot \vec{w}$$

$\vec{v} = \begin{pmatrix}1\\2\end{pmatrix}, \quad \vec{w} = \begin{pmatrix}3\\4\end{pmatrix}$

$D(\vec{v},\vec{w}) = D(1\vec{e_1} + 2\vec{e_2},\ 3\vec{e_1} + 4\vec{e_2})$

$= D(\vec{e_1},\ 3\vec{e_1} + 4\vec{e_2}) + D(2\vec{e_2},\ 3\vec{e_1} + 4\vec{e_2})$

$= D(\vec{e_1},\ 3\vec{e_1} + 4\vec{e_2}) + 2D(\vec{e_2},\ 3\vec{e_1} + 4\vec{e_2})$

$= D(\vec{e_1},\ 3\vec{e_1}) + D(\vec{e_1},\ 4\vec{e_2})$

$+ 2D(\vec{e_2},\ 3\vec{e_1}) + 2D(\vec{e_2},\ 4\vec{e_2})$

$= 3\cancelto{1}{D(\vec{e_1},\vec{e_1})} + 4\cancelto{0}{D(\vec{e_1},\vec{e_2})}$

$+ 2\cdot 3\cancelto{0}{D(\vec{e_2},\vec{e_1})} + 2\cdot 4\cancel{D(\vec{e_2},\vec{e_2})},$

$= 1\cdot 3 + 2\cdot 4$

$= 11$

$\begin{pmatrix}1\\2\end{pmatrix} \cdot \begin{pmatrix}3\\4\end{pmatrix} = 1\cdot 3 + 2\cdot 4 = 1$

$$\underbrace{\vec{v}\cdot\vec{w}}_{\text{alg}} = \underbrace{\|\vec{v}\|\,\|\vec{w}\|\cos\theta}_{\text{geom}}$$

synthesized

$\vec{e}_i \mapsto \vec{v}_i$

$$\vec{v}_i\cdot\vec{v}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

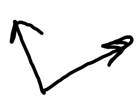
orthonormal basis

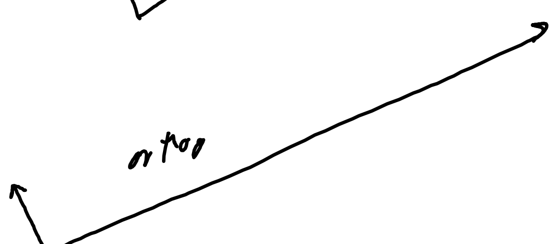
orthogonal

$$\vec{v}_i\cdot\vec{v}_j = \begin{cases} \text{nonzero} & i=j \\ 0 & i\neq j \end{cases}$$

$$\frac{\vec{v}_i}{\|\vec{v}_i\|}$$

orthonormal





4. $\text{proj}_L(\vec{x})$

$\vec{u}$ nonzero in L

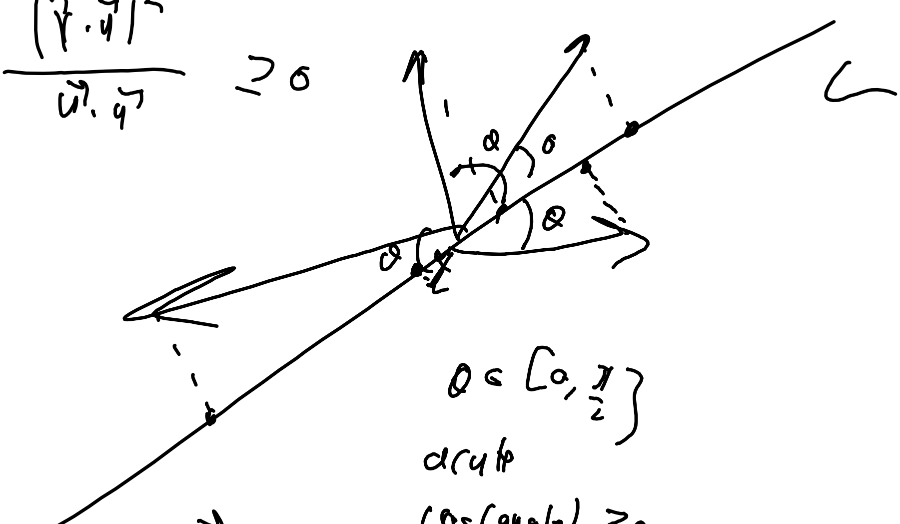
$\vec{u}$ spans L

$$\text{proj}_L(\vec{x}) = \frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

$$\vec{x}\cdot\left(\frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}\right)$$

$$\frac{(\vec{x}\cdot\vec{u})}{\vec{u}\cdot\vec{u}}\,(\vec{x}\cdot\vec{u})$$

$$\frac{(\vec{x}\cdot\vec{u})^2}{\vec{u}\cdot\vec{u}} \geq 0$$



$\theta \in [0, \frac{\pi}{2}]$

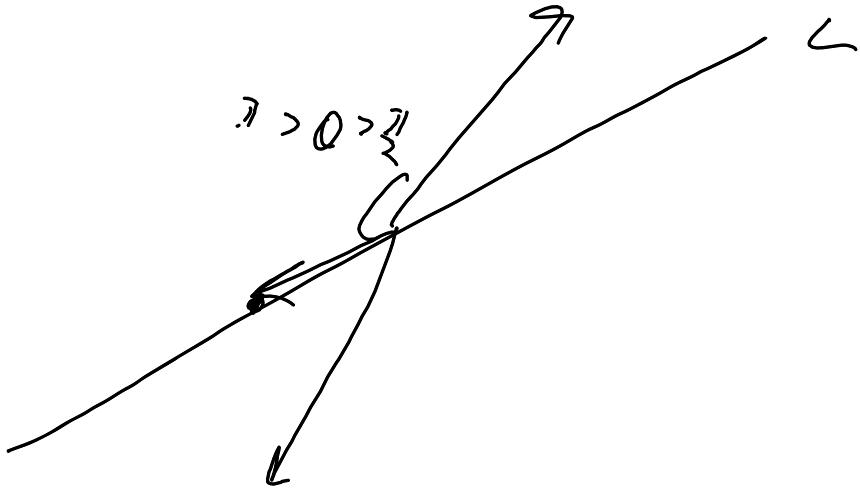
acute

$\cos(\text{angle}) \geq 0$

$\vec{x}\cdot\text{proj}_L(\vec{x})$

$= \|\vec{x}\|\,\|\vec{v}\|\cos(\text{acute}) \geq 0$

$\vec{x} \cdot \text{proj}_L(-\vec{x})$



$$-\underbrace{\vec{x} \cdot \text{proj}_L(\vec{x})}_{\geq 0}$$