wksht 4

doable: 2, 3   1?

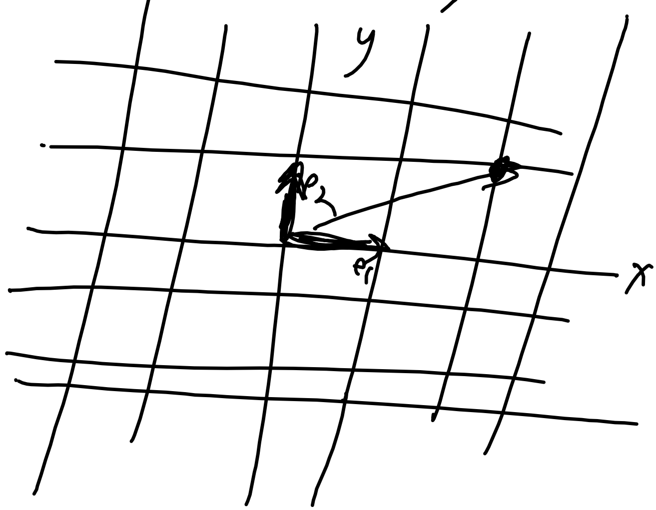
4, 5   challenge

- projections ✓
- reflections ✓
- rotations ✓
- shears ✓

} linear

2d play

2d plane w/a grid/axes



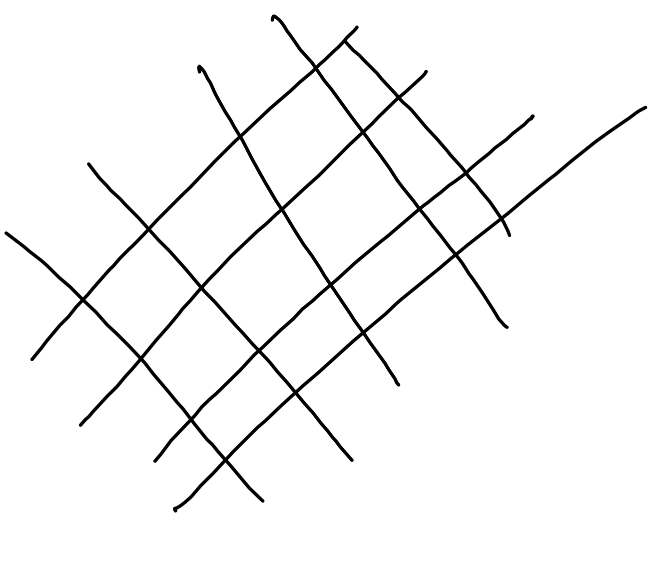
$2\vec{e_1} + 1\vec{e_2}$

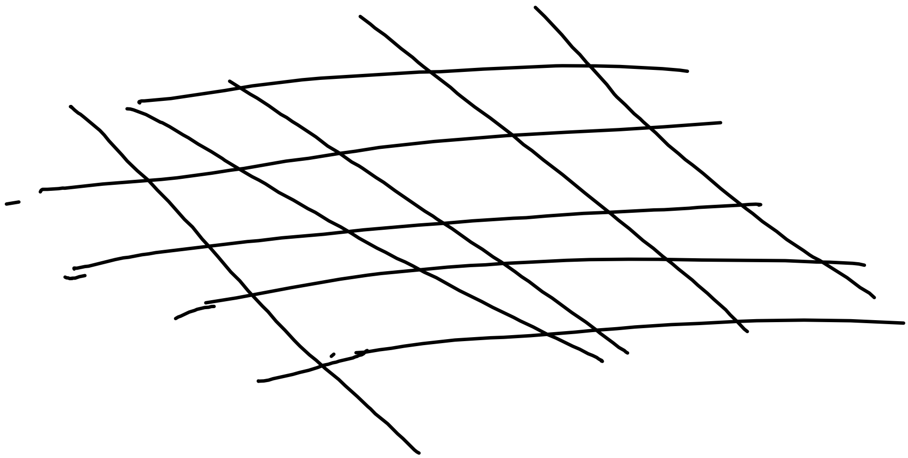
Transforming the grid $\longleftrightarrow$ matrices

$T(\vec{x}) = A\vec{x}$ formula

A has i^{th} colum $T(\vec{e_i})$

$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ i^{th} pos

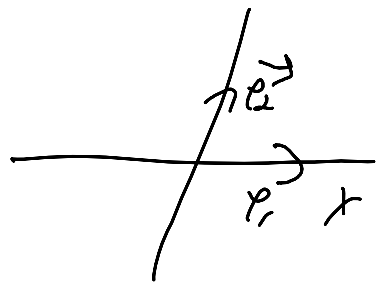




$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

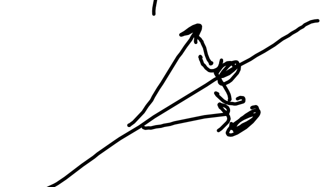
reflection about x-axis

matrix? (in standard basis) $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$



$T(\vec{e_1}) = \vec{e_1} = \begin{pmatrix}1\\0\end{pmatrix}$

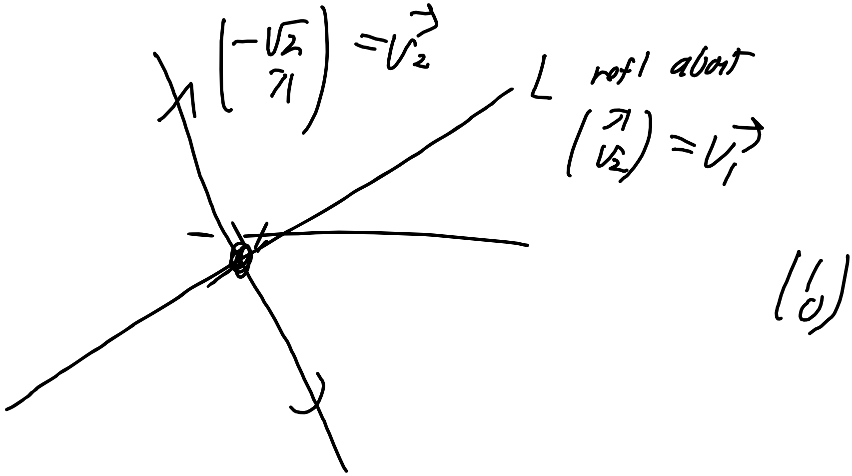
$T(\vec{e_2}) = -\vec{e_2} = \begin{pmatrix}0\\-1\end{pmatrix}$



reflection about the line spanned by $\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$.

$T(\vec{e_1}) = ?$

$T(\vec{e_2}) = ?$



$$T(\vec{v_1}) = \vec{v_1}$$
$$T(\vec{v_2}) = -\vec{v_2}$$

$$\mathcal{B} = (\vec{v_1}, \vec{v_2})$$

$$[T]_{\mathcal{B}} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Moral. first find a basis that respects the geometry

reflections about a line

rotations about a line

projection onto a line

perpendicular

$$\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\begin{pmatrix}1\\2\end{pmatrix}_{\mathcal{E}}$$

$$\begin{pmatrix}-1\\-1\end{pmatrix}_{\beta'''}$$

$$\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}_{\beta''}$$

$$\begin{pmatrix}1\\0\end{pmatrix}_{\beta} \qquad \begin{pmatrix}0\\1\end{pmatrix}_{\beta'} \qquad \begin{pmatrix}\pi\\ \sqrt{2}\end{pmatrix}_{\beta''}$$

before: lin trans $\longleftrightarrow$ matrices

$\downarrow$ standard basis

now: lin trans + a basis $\longleftrightarrow$ matrices

$$\vec{v}_1 = \begin{pmatrix}\pi\\ \sqrt{2}\end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix}-\sqrt{2}\\ \pi\end{pmatrix}$$

$$\beta = (\vec{v}_1, \vec{v}_2)$$

$$[T]_{\beta} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$[T]_{\mathcal{E}}\,\vec{v}_1 = \vec{v}_1$$

$$[T]_{\mathcal{E}}\,\vec{v}_2 = -\vec{v}_2$$

S inv. $\qquad A = S^{-1}$

$A^{-1}[T]_\beta A = [T]_\gamma$

$$\left[S\,[T]_\beta\,(S^{-1}) = [T]_\gamma \right]$$

$$S\,[T]_\beta\,S^{-1} = [T]_\gamma$$

find S?

$S = {}_\gamma P_\beta$

know $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

want to know $[T]_\gamma = ?$

we need to know S

$$S = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$$

$$\beta \xrightarrow{S} \gamma$$

$\mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}_\beta \qquad S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}_\gamma$

$$\boxed{S = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & & & | \end{pmatrix}}$$ $\mathcal{B}$ to $\mathcal{E}$

$$\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$$

$$\mathcal{B}' = (\vec{v}_2, \vec{v}_1) \qquad [T]_{\mathcal{B}'} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S [T]_{\mathcal{B}} S^{-1} = [T]_{\mathcal{E}}$$

Prob 2. find S 2x2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \quad S^{-1} \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} S = \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Some b, d

gvm | abstract reasoning

$$S = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$S^{-1} = \frac{1}{xw - yz} \begin{pmatrix} w & -y \\ z & x \end{pmatrix}$$

$$\frac{1}{xw - yz} \begin{pmatrix} w & -y \\ z & x \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix}$$

$\begin{matrix} x & y \\ z & w \end{matrix}$

$$S [T]_\beta S^{-1} = [T]_\delta \quad (= (\vec{e}_1, \vec{e}_2))$$

$$S = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} \qquad \beta = (\vec{v}_1, \vec{v}_2)$$

$$[T]_\beta S^{-1} = S^{-1} [T]_\delta$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = [T]_\delta$$

$$\begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$$

$$[T]_\beta = S^{-1} [T]_\delta S$$

$$[T]_\delta = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{tikzcd} \beta \arrow[r, "S"', "\text{translate}"] & \delta \arrow[d, "{[T]_\delta}"] \\ \beta & \delta \arrow[l, "S^{-1}", "\text{translate}"'] \end{tikzcd}$$

Find a basis β s.t.

$$[T]_\beta = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$$

$\beta = (\vec{v}_1, \vec{v}_2)$

$[T]_\beta = (\dots)$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_\beta \longleftrightarrow \vec{v}_1$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in β is $\vec{v}_1$

def

$$[T(\vec{v}_i)]_\beta = [T]_\beta \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$[T(\vec{v}_1)]_{\mathcal{B}} = \underline{[T]_{\mathcal{B}}}\ \underline{\begin{pmatrix}1\\0\end{pmatrix}}$$

$\vec{v}_1$ in $\mathcal{B}$

$$\begin{pmatrix}a\\1\end{pmatrix} \qquad \begin{pmatrix}a & b\\1 & d\end{pmatrix}$$

$$[T(\vec{v}_1)]_{\mathcal{B}} = \begin{pmatrix}a\\1\end{pmatrix}$$

which vector $\vec{w}$ has $\begin{pmatrix}c\\1\end{pmatrix}$ as its $\mathcal{B}$-repr.

$\vec{w} = \vec{v_2}$

$T(\vec{v}_1) = \vec{v_2}$

find $\vec{v}_1, \vec{v}_2$ s.t. $\begin{pmatrix}a & b\\1 & d\end{pmatrix}$ in $\mathcal{B}$

need $\underline{T(\vec{v}_1)} = \underline{\vec{v_2}}$ — lin indep

$$\begin{pmatrix}1 & 3\\2 & 4\end{pmatrix} = [T]_{\mathcal{S}}$$

find invart $\rightarrow$ $\vec{v_1}, \vec{v_2}$

lin indep

$T(\vec{v_1}) = \vec{v_2}$

$[T]_{\cancel{\beta}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\vec{v_1} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\rightarrow$ ~~$T(\vec{v_1}) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$~~

$T(\vec{v_1}) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\left(\vec{v_1},\ T(\vec{v_1}) \right)$

inv.

$S = \begin{array}{cc} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \\ \vec{v_1} \quad \vec{v_2} \end{array}$

$T(\vec{v_1}) = \vec{v_2}$

$[T]_B = \begin{pmatrix} 0 & * \\ 1 & * \end{pmatrix}$

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$xyzy$ ——— $= 0$

$xyzw$ ——— $= 1$

SAS^{-1}

$\neq$

$S^{-1}AS$

$P = S^{-1}$

$P[T]_\beta P^{-1}$

$B = S^{-1}$

$=$

$B^{-1}AB$

"the"

$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$

$S^{-1}[T]_\beta S = [T]_\gamma$

$S \quad \beta \longrightarrow \gamma$ "the" change of basis matrix

$P \quad \gamma \longrightarrow \beta$ "the" — — —

$P = S^{-1}$

$$S = {}_{\underline{\alpha}}[I]_{\underline{\beta}}$$

$$S^{-1} = {}_{\beta}[I]_{\alpha}$$

$$S \; {}_{\beta}[T]_{\beta} \; S^{-1}$$

$$= {}_{\alpha}[I]_{\beta} \; {}_{\beta}[T]_{\beta} \; {}_{\beta}[I]_{\alpha}$$

$${}_{\alpha}[T]_{\alpha}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = [T]_\beta \begin{pmatrix}1\\0\end{pmatrix}$$

$T(\vec x) = A\vec x$

$A = [T]_{\mathcal{E}}$

$\boxed{T(\vec v_1) = \vec v_2}$

$[\vec v_1]_\beta = \begin{pmatrix}1\\0\end{pmatrix}$

$\beta = (\vec v_1, \vec v_2)$

$[T]_\beta \, [\vec v_1]_\beta$

$=$

$[T(\vec v_1)]_\beta$

$=$

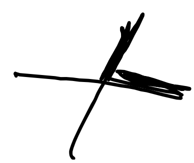
$\begin{pmatrix}0\\1\end{pmatrix} = [v_2]_\beta$

$\beta = (\vec v_1, \vec v_2)$

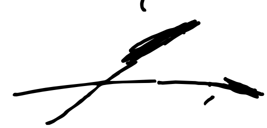
$\begin{pmatrix}a\\b\end{pmatrix} = a\vec e_1 + b\vec e_2$

$= a\vec v_1 + b\vec v_2$

context $\mathcal{E}$



context β



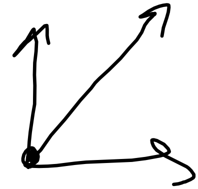
$\begin{pmatrix}a\\b\end{pmatrix} = [\vec w]_\beta$

$\vec w = a\vec v_1 + b\vec v_2$

$[v_2]_\beta = \begin{pmatrix}0\\1\end{pmatrix}$

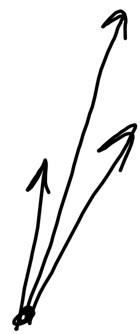
$\begin{pmatrix}0\\1\end{pmatrix}$ in β $= 0\vec v_1 + 1\vec v_2 = \vec v_2$

$$\begin{pmatrix}1\\1\end{pmatrix} = [\vec{w}]_{\mathcal{B}}$$



$$\vec{w} = \vec{v_1} + \vec{v_2}$$

$$\begin{pmatrix}1\\1\end{pmatrix} \quad \begin{pmatrix}1\\1\end{pmatrix}$$



$$\begin{pmatrix}1 & b\\1 & d\end{pmatrix}\vec{e_1} = \vec{e_1}+\vec{e_2} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$T(\vec{v_1}) = \vec{v_1} + \vec{v_2}$$

$$\underbrace{[T]_{\mathcal{B}}}_{\text{matrix}} [v]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{B}}$$

$$A = [T]_{\mathcal{B}}$$

$$T(\vec{v}) = A\vec{v}$$

$$[T(\vec{v})]_{\cancel{\mathcal{A}}\,\mathcal{B}} = [T]_{\cancel{\mathcal{A}}\,\mathcal{B}} [\vec{v}]_{\cancel{\mathcal{A}}\,\mathcal{B}}$$