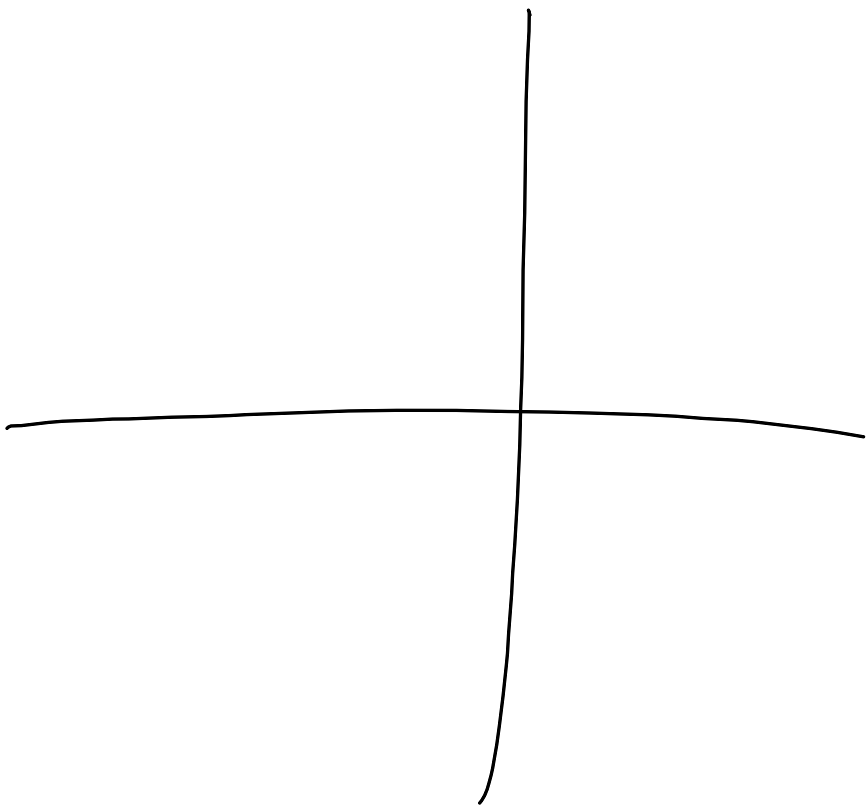
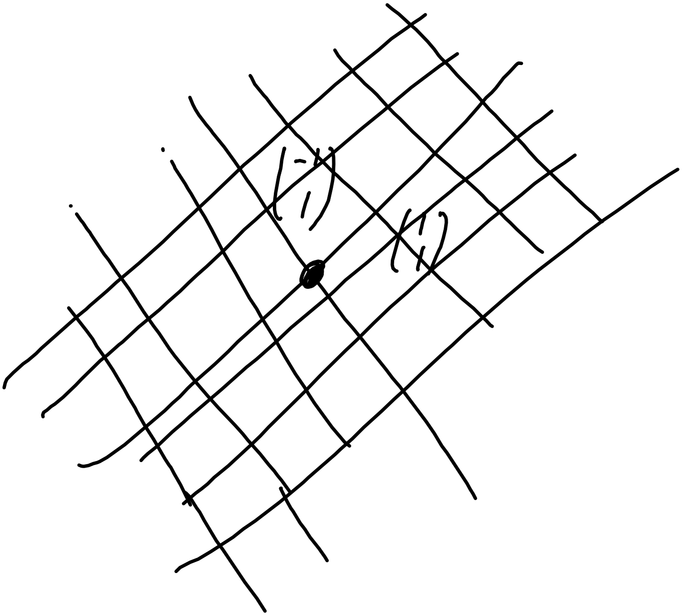
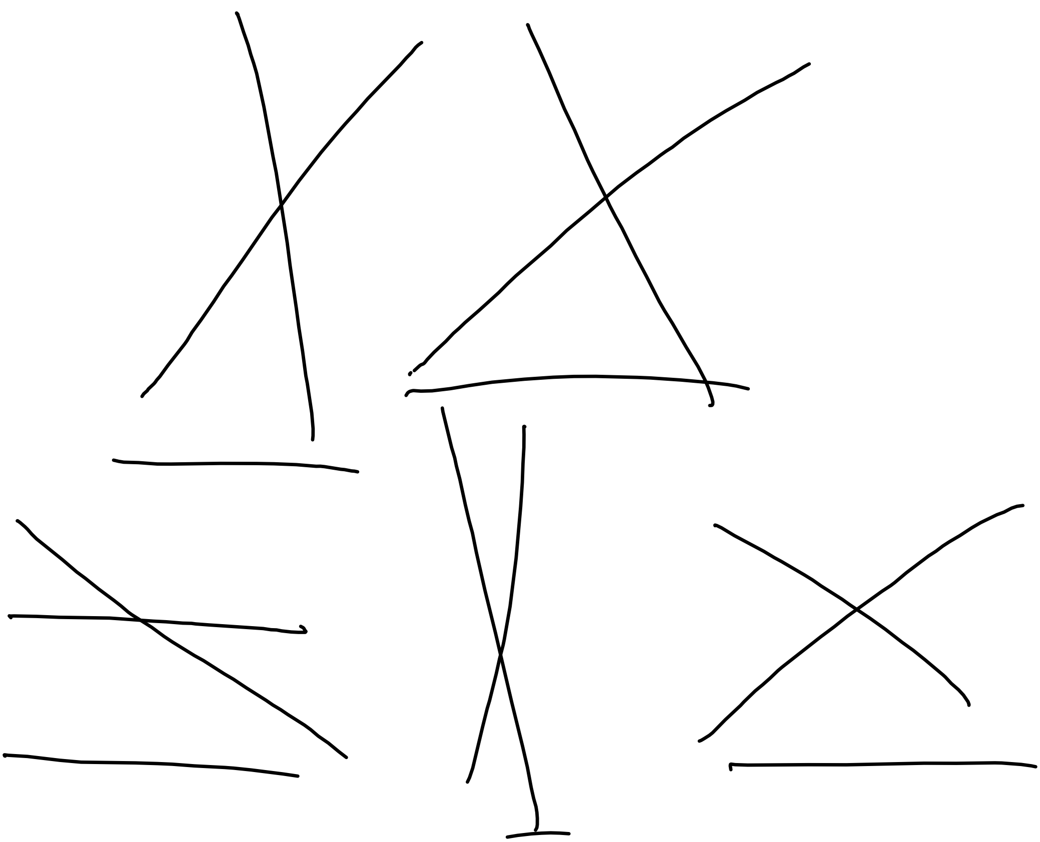
2d spare

2d space, w/ axes

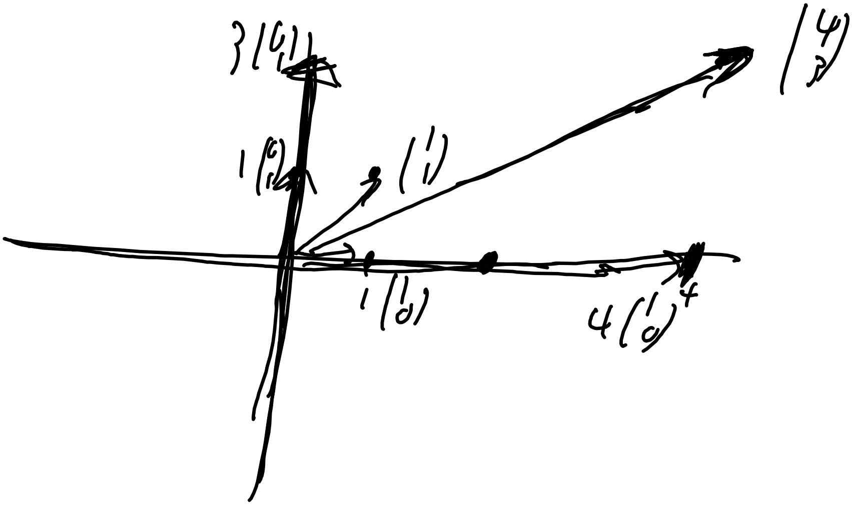


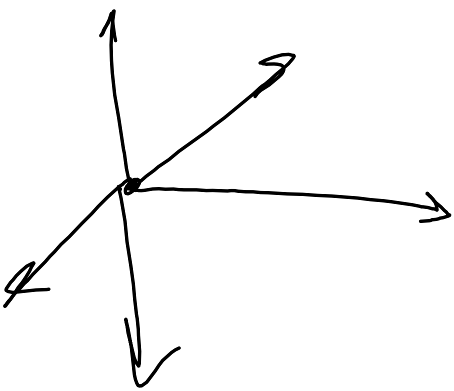
diff. axes



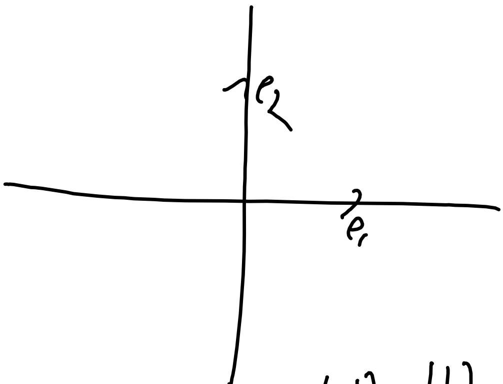


$$\underline{\begin{pmatrix} x \\ y \end{pmatrix}} = x\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

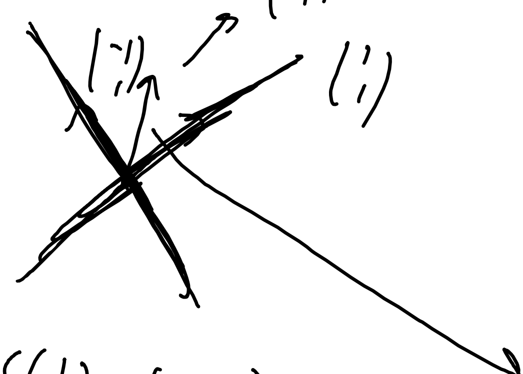




$\vec{e_1}, \vec{e_2}, \vec{e_3}, \ldots, \vec{e_4}$



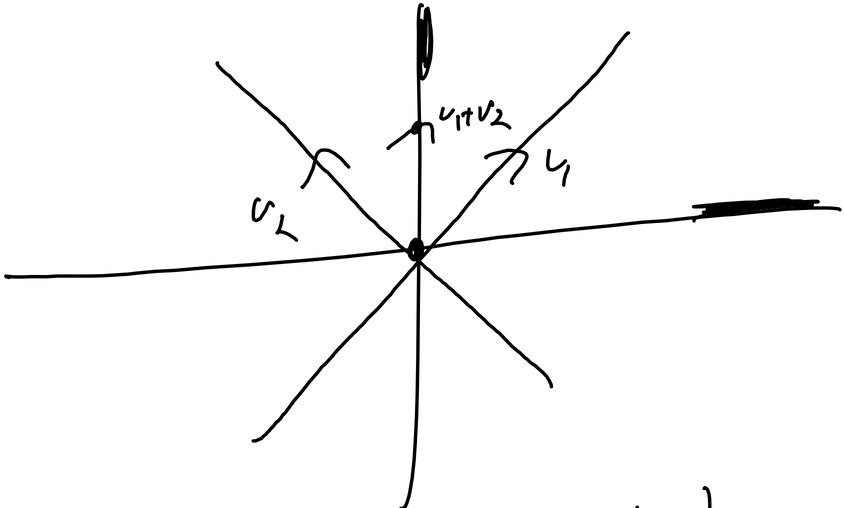
$\begin{pmatrix}-1\\1\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix} = \vec{v_1}+\vec{v_2}$



$$\beta = \left\{ \underset{\vec{v_1}}{\begin{pmatrix}1\\1\end{pmatrix}}, \underset{\vec{v_2}}{\begin{pmatrix}-1\\1\end{pmatrix}} \right\}$$

$$\begin{pmatrix}1\\1\end{pmatrix}_\beta = \vec{v_1}+\vec{v_2}$$

$$\begin{pmatrix}1\\0\end{pmatrix}_\beta = \vec{v_1} \qquad \begin{pmatrix}1\\3\end{pmatrix}_\beta = 1\vec{v_1} + 3\vec{v_2}$$



$\text{name}_1 : \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\text{name}_2 : \begin{pmatrix} 1 \\ 1 \end{pmatrix}_\beta = 1\vec{v_1} + 1\vec{v_2}$

β is called a basis

change of coordinates

standard coords $\dagger$, $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

β-coords $\times$, $\boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}_\beta}$

$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathcal{B} = (\vec{v_1}, \dots, \vec{v_n})$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{B}} = a_1\vec{v_1} + a_2\vec{v_2} + \dots + a_n\vec{v_n}$$

in $\mathcal{B}$ coordinats

$$\mathcal{S} = (\vec{e_1}, \dots, \vec{e_n})$$

"standard basis"

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{S}} = a_1\vec{e_1} + \dots + a_n\vec{e_n}$$

$$= a_1\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + a_n\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}}$$

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear

find a matrix A representing T

$$T(\vec{x}) = A\vec{x}$$

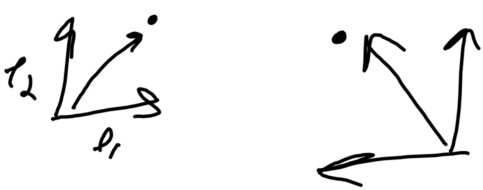
How do we find A?

$T(\vec{e}_1)$ first col of A

$T(\vec{e}_2)$ second col of A

$\vdots$

$T(\vec{e}_i)$ i^{th} col of A



$T(\vec{x}) = A\vec{x}$ in $\mathcal{E}$

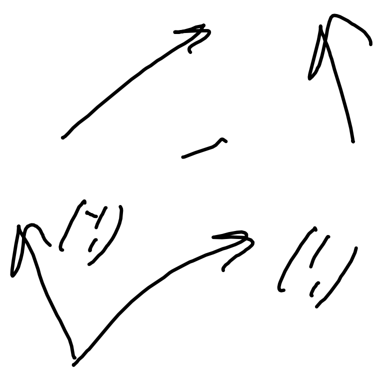
e_i gets the columns

$\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ basis

$[T]_{\mathcal{B}}$ is a matrix, $n \times n$ such that

$$[T]_{\mathcal{B}} \cdot [\vec{v}]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{B}}$$

$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$



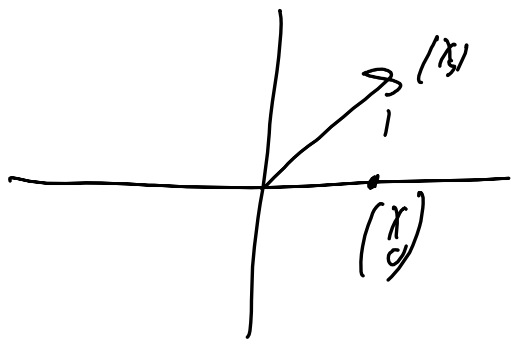
$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

projection onto $\{y = x\}$.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

proj onto x-axis?

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$



much easier

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_\beta = \vec{v}_1$

line

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \{y = x\}$

$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

proj T onto $\{y=x\}$ in this basis?

matrix A represents T in this basis,

$$[T(\vec{x})]_\beta = A\,[\vec{x}]_\beta$$

$$[T(\vec{v}_1)]_\beta = A\,[\vec{v}_1]_\beta = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

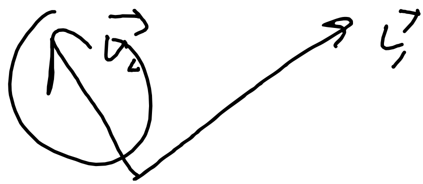
$T(\vec{v}_1) = \vec{v}_1$. $\rightarrow$ $[\vec{v}_1]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$A\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

$$A\,[\vec{v}_2]_\beta = [T(\vec{v}_2)]_\beta$$

$$A\begin{pmatrix}0\\1\end{pmatrix} \qquad [\vec{0}]_\beta=\begin{pmatrix}0\\0\end{pmatrix}$$

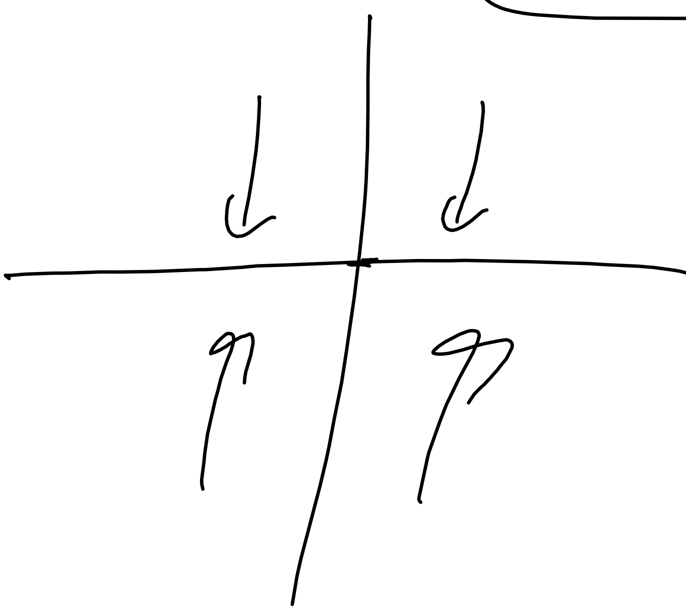
$$A\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

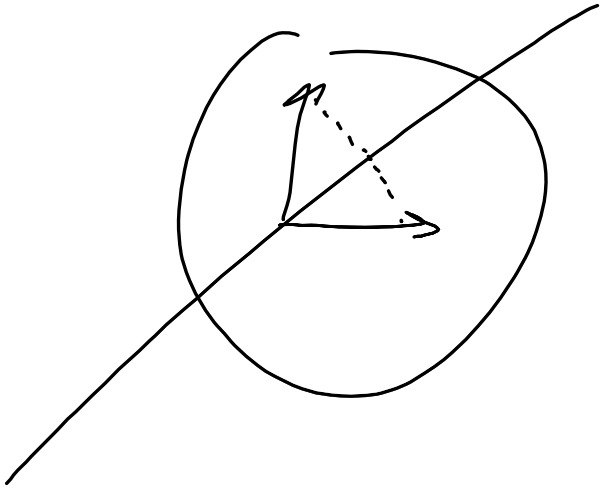


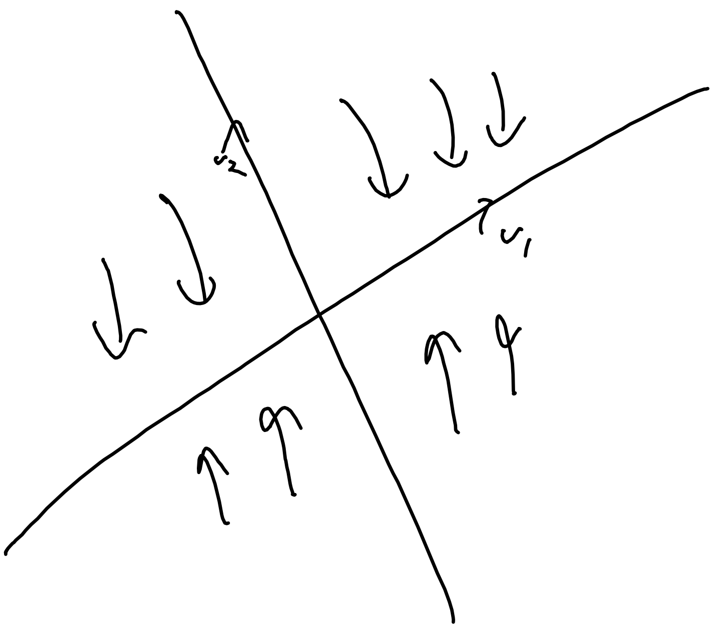
$$A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$T(\vec{v}_2)=0$

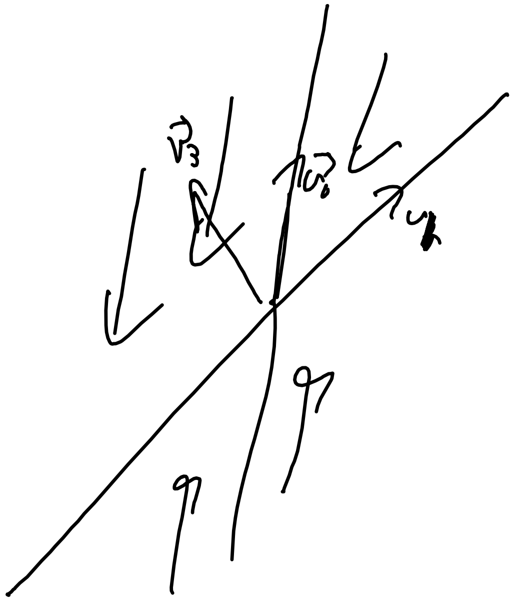
in the new basis







v_2
v_1



$\mathcal{B} = (\vec{v}_1, \vec{v}_2)$

span q

proj matrix onto line $\vec{v}_1$

along $\vec{v}_2$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathcal{B}$

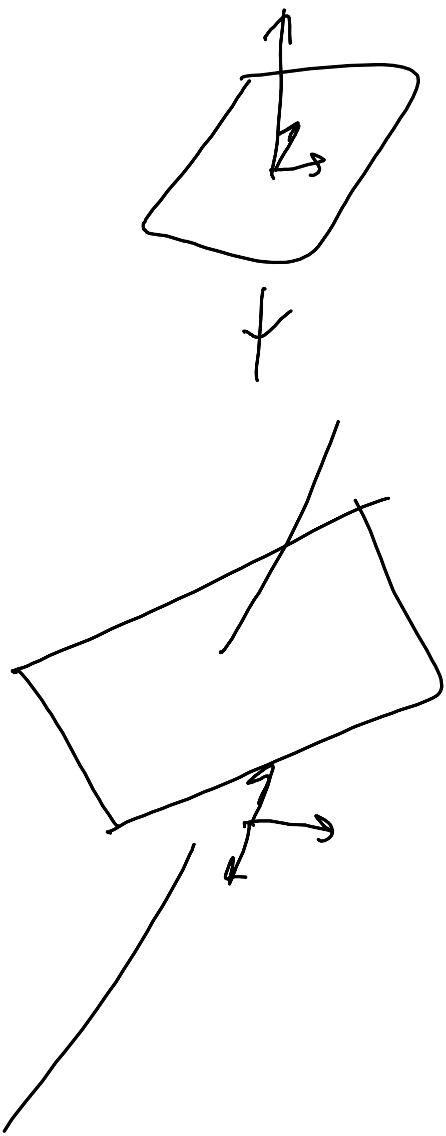
(v_1, v_3) bad

(v_1, v_2) good

food for thought.

anyway,

projection onto some angled plane



Spray $\left(\left(\frac{1}{\sqrt{2}}\right)\right)$

$$[\vec{v}]_\beta \longleftrightarrow [\vec{v}]_\mathcal{S} \qquad \beta = (\vec{v_1}, \vec{v_2}, \dots \vec{v_n})$$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_\mathcal{S} \longleftrightarrow \sum a_i \vec{e_i}$$

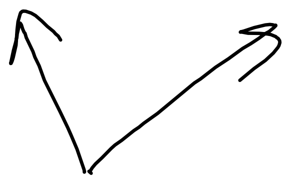
$$\beta \longleftrightarrow \sum a_i \vec{v_i}$$

$$S = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}$$

$$S \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_\mathcal{S}}_{\text{coeff of lin combo of cols of } S} = \underline{\sum a_i \vec{v_i}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_\beta$$

$$\underbrace{S}_{\text{translator}} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_\mathcal{S} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_\beta$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$



$$S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\beta}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_1$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\beta}$$

$$\underbrace{{}_{\beta}[I]_{\delta}}\,[\vec{v}]_{\delta}$$

$$= [\vec{v}]_{\beta}$$

$$ {}_{\beta}[I]_{\delta} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\delta}$$

$$\shortparallel \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\beta}$$

I written

$$\underbrace{{}_{\delta}[I]_{\beta}}\ \underset{T\ \beta\text{ coords}}{\underline{{}_{\beta}[T]_{\beta}}}\ \underbrace{{}_{\beta}[I]_{\delta}} = \underset{T\ \delta\text{ coords}}{\underline{{}_{\delta}[T]_{\delta}}}$$

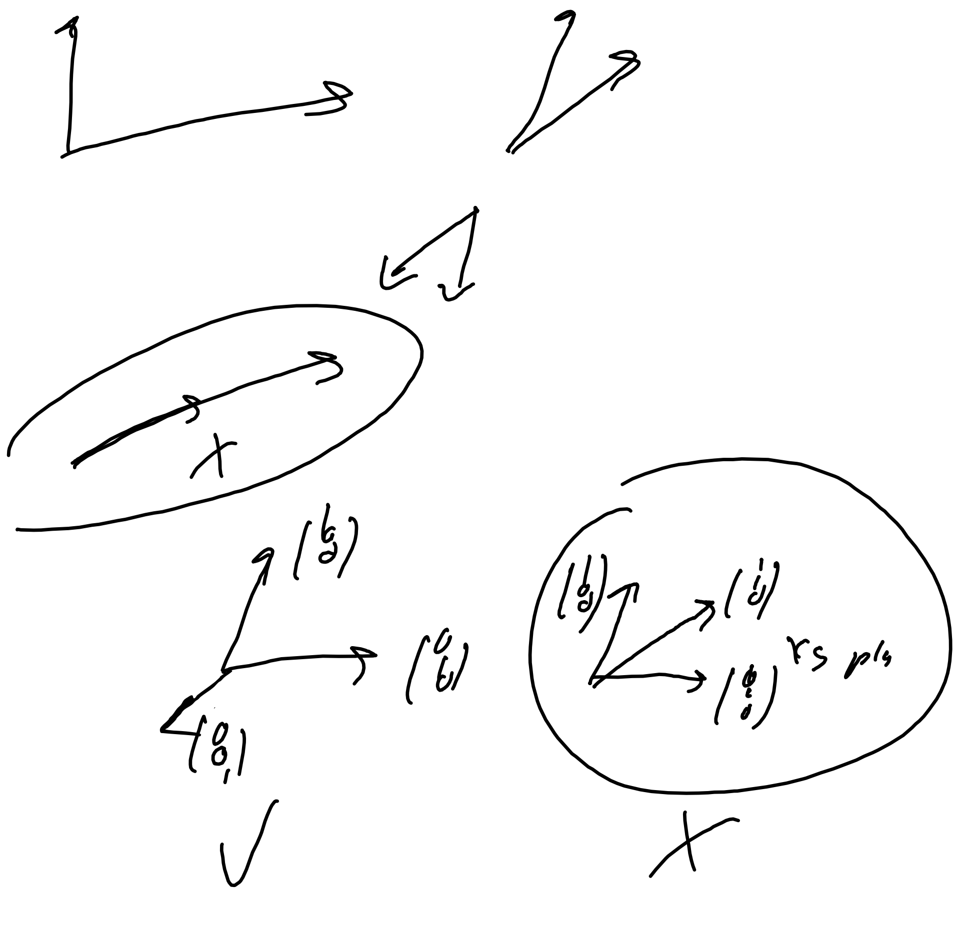
$\beta \longmapsto \delta$ $\quad$ $\delta \longmapsto \beta$

cols $\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$

$\vec{v}_1, \dots, \vec{v}_p$ lin. indep in V

$\vec{w}_1, \dots, \vec{w}_q$ span V

claim: $q \geq p$



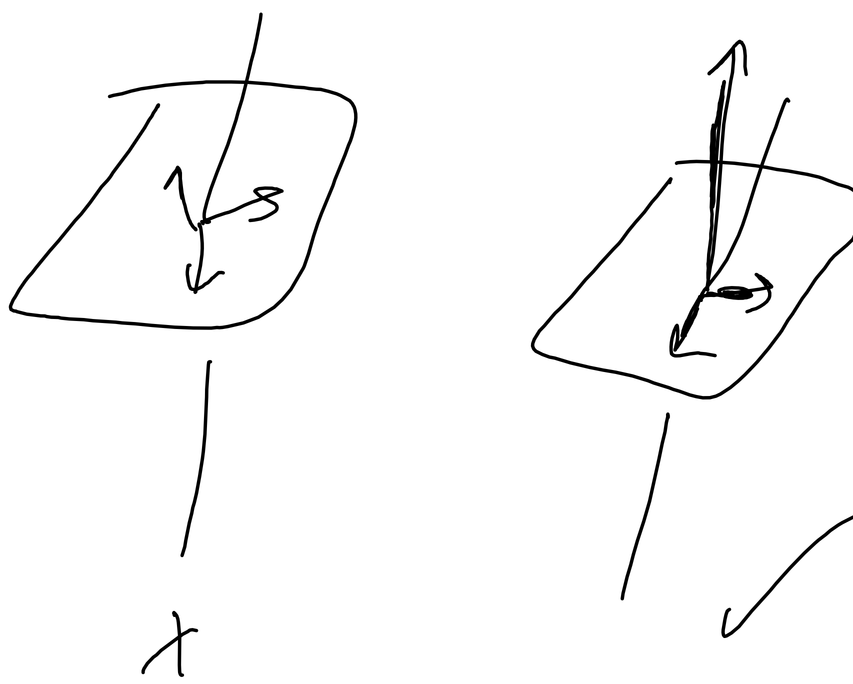
$\vec{v}_1, \ldots \vec{v}_p$ not lin indep

$$\sum_{i=1}^{p} a_i \vec{v}_i = 0$$

$a_i \neq 0$

geometrically: $v_1, \ldots, v_p$ all in some

$< p$ dim span



$\vec{v}_1, \dots, \vec{v}_p$ lin indep

> axes for a p-dim space
> no p-1 dim space containing them all.

$\vec{w}_1, \dots, \vec{w}_q$ span V.

$$A = \begin{pmatrix} | & & | \\ \vec{w}_1 & \cdots & \vec{w}_q \\ | & & | \end{pmatrix}$$

rank $= \dim V$

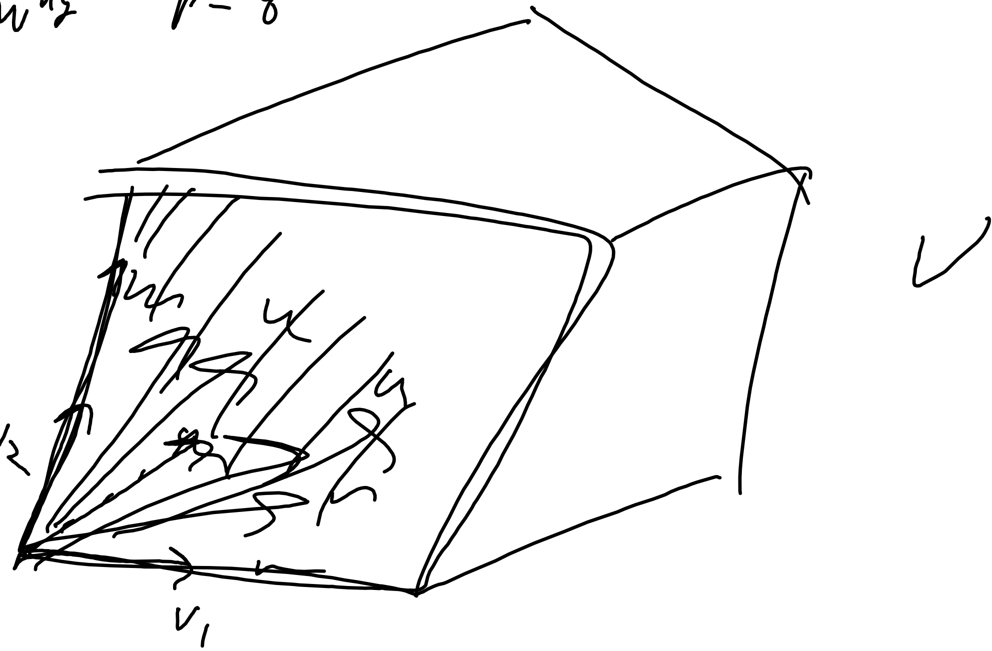
every $\vec{x}$ in V looks like

$\vec{x} = a_1 \vec{w}_1 + \dots + a_q \vec{w}_q$, a lin comb

$\mathrm{im}(A) = V$

$q \geq p$?

w^{hy} $p \le q$



v_1, v_2 base for a subspace of V

nullity $\leftrightarrow$ linear transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

null(T) ker(T)

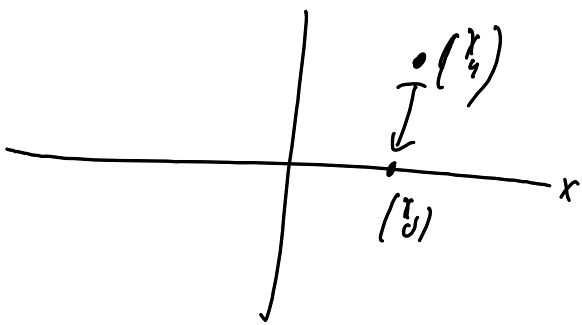
nullspace   kernel

|| //

solutions to $T(\vec{x}) = \vec{0}$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ proj onto x-axis

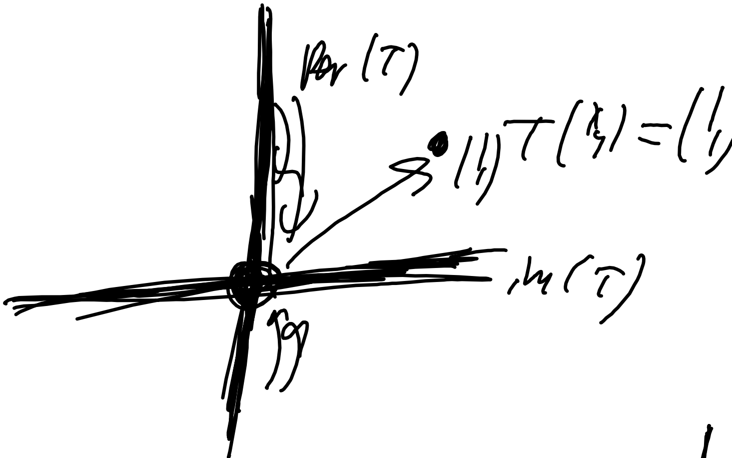
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

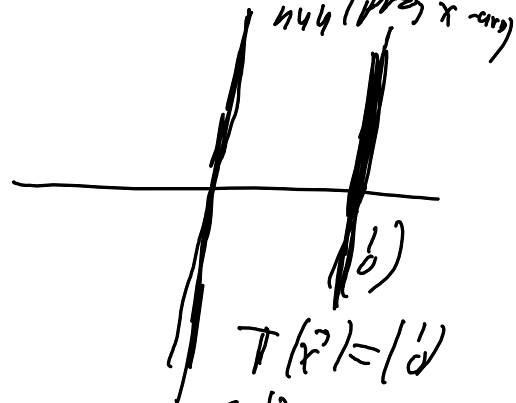


null(T)
ker(T) = y-axis $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x = 0$





$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{T} & \mathbb{R}^m \\ \cup & & \cup \\ \ker & & \text{im} \\ \text{null} & & \end{array}$$

$T\begin{pmatrix} x \\ ? \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\{(1,0)\}$ $\begin{pmatrix} 1 \\ y \end{pmatrix}$

$y \in \mathbb{R}$

sols many

constrained

rank-nullity thm

given $T: \mathbb{R}^n \to \mathbb{R}^m$ linear

$$\text{Rank}(T) + \text{nullity}(T) = n$$

$\text{Rank}(T) = \dim \text{im}\, T$ = # 1s rref(A), A repr T

$\text{nullity}(T) = \dim \ker(T) = \dim \text{null}(T)$

$\text{im}(T) \subseteq \mathbb{R}^m$

$\ker(T) \subseteq \mathbb{R}^n$

$T(\vec{x}) = \vec{y}$? find $\vec{x}$ to make this true

I found $\vec{a}$ so that $T(\vec{a}) = \vec{y}$

what are all the solutions

$T(\vec{x}) = \vec{0}$

$\vec{0} = \vec{0}$

$T(\vec{0}) = \vec{0}$

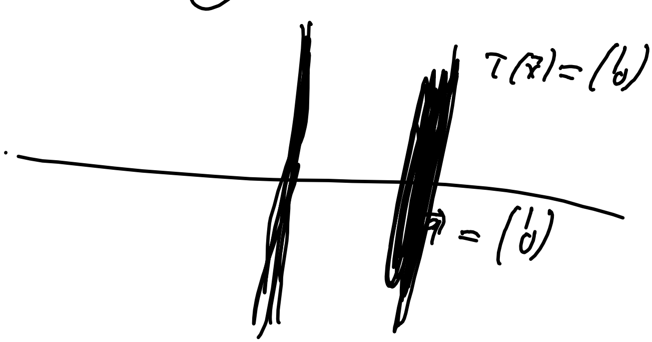
$\ker(T)$

another solution $\vec{b}$

$$T(\vec{a}) = T(\vec{b}) = \vec{y}$$

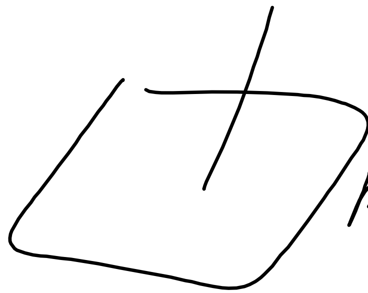
$$T(\vec{a} - \vec{b}) = \vec{0}$$

$\vec{a} - \vec{b} \in \ker(T)$

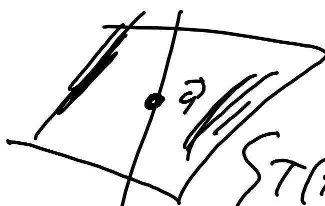


$T\begin{pmatrix}1\\c\end{pmatrix} = \begin{pmatrix}1\\c\end{pmatrix}$

T proj. x-axis



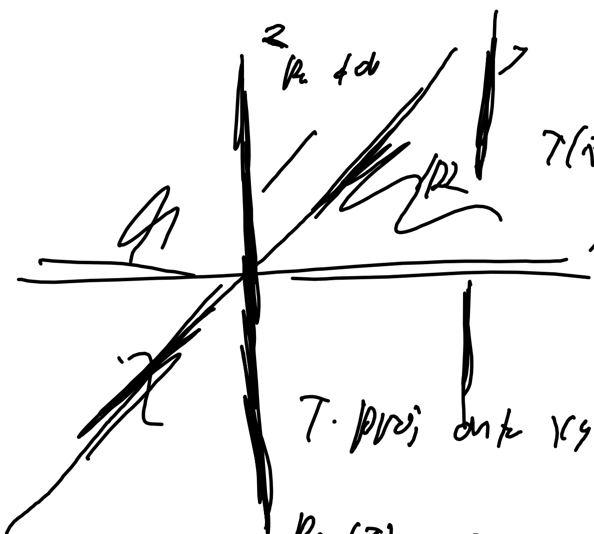
$\ker(T)$ $\{T\vec{x} = 0\}$



$T(x) = \vec{y}$

$\{T(\vec{x}) = \vec{y}\}$

$\mathbb{R}^3$



$T(\vec{x}) = (\ldots)$

T: proj onto xy-plane

$\ker(T) = z$ axis

$\operatorname{im}(T) \cap \ker(T) = 0$

$T: \mathbb{R}^3 \longrightarrow \underline{\mathbb{R}}$ $\vec{v} \in \mathbb{R}^3$

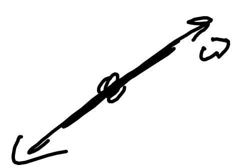
$$T(\vec{w}) = \vec{v} \cdot \vec{w}$$

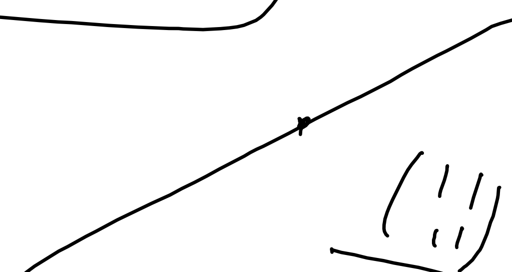
$$\vec{v} \cdot (\vec{w}_1 + \vec{w}_2)$$
$$= \vec{v} \cdot (\vec{w}_1) + \vec{v} \cdot \vec{w}_2$$

linear!

basis for ker, im

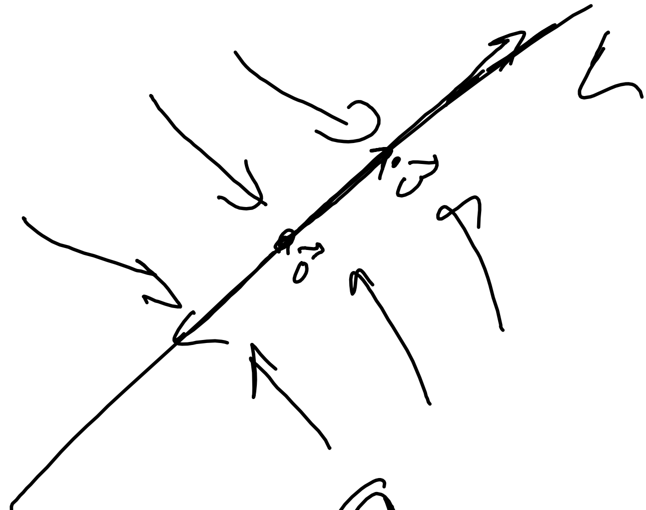
$$\mathrm{proj}_{\vec{v}}(\vec{w}) = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$





$$\frac{\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}}{\sqrt{2}}$$

$\mathrm{proj}_L \vec{w}$



$\vec{v} \neq 0$

$\vec{v}$ unit vector

$\|v\| = 1$

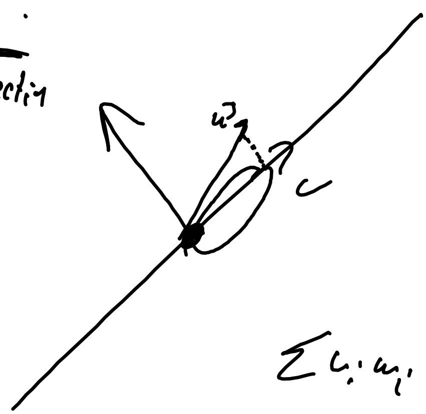
$\underline{c}\,\underline{\vec{v}}$ $c \in \mathbb{R}$ unique

$(\vec{v} \cdot \vec{w})\,\underline{\vec{v}}$:

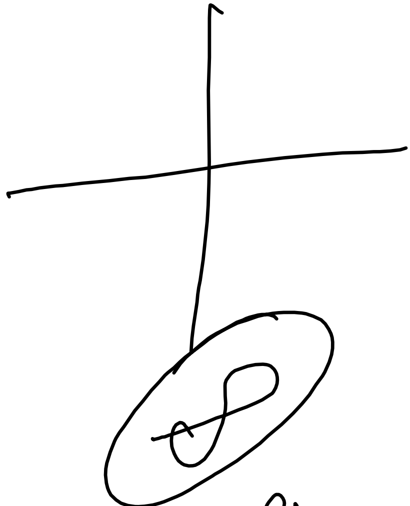
direction

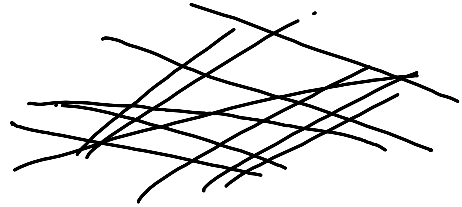
how much of $\vec{w}$ lives in $\vec{v}$ direction

measurement of correlation



$\sum v_i w_i$





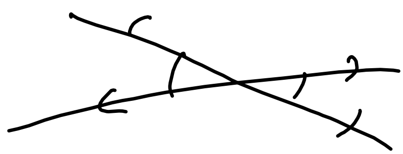
β

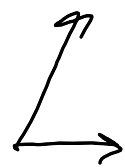
$\underline{e_i} \longleftrightarrow \underline{v_i}$

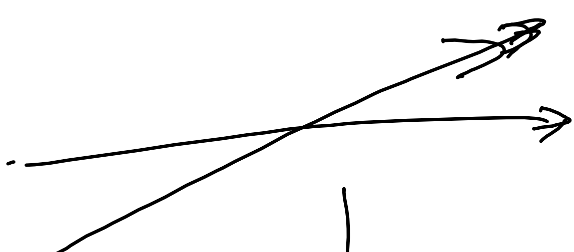
$v_i \cdot v_j = 0 \quad i \neq j$

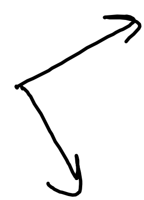
$\|v_i\| = 1$

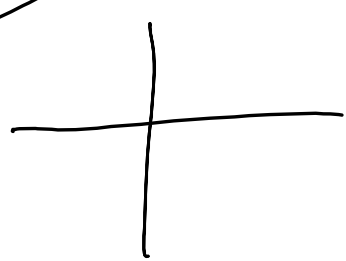
$$\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$$

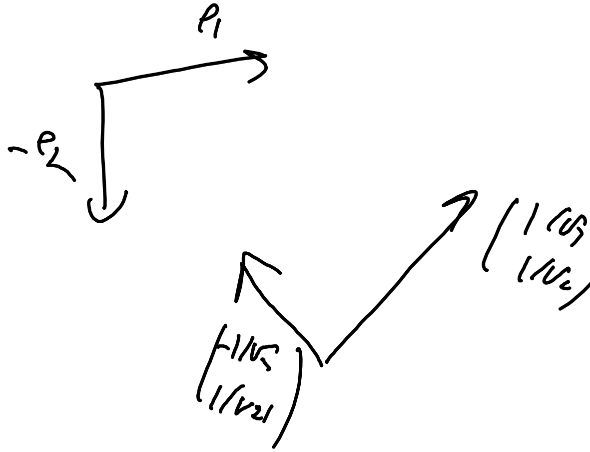












$\mathcal{B} = (\vec{v_1}, \vec{v_n})$

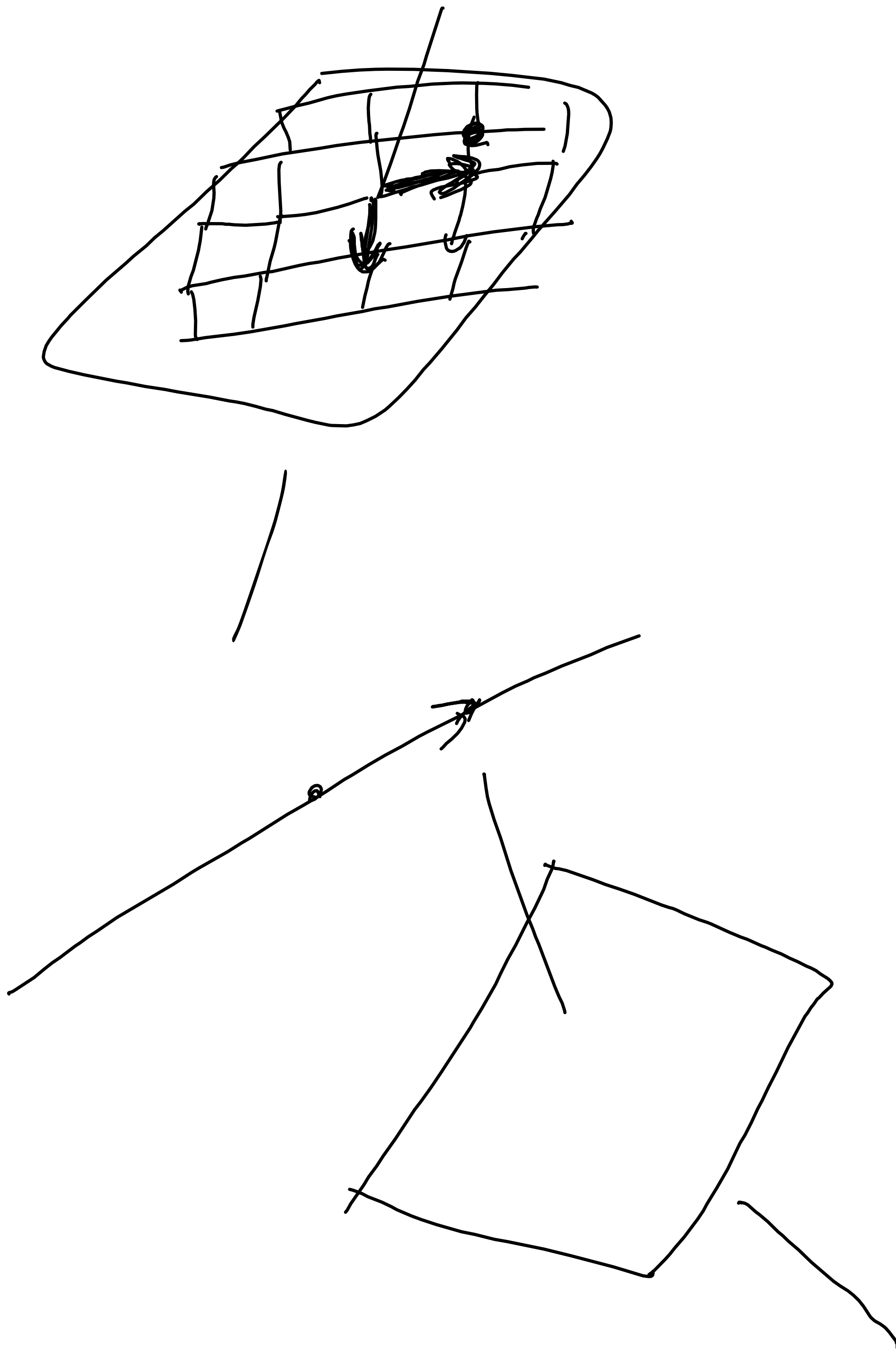
span ← every vector $\vec{v}$ is a lin combo of $\vec{v_i}$

lin indep ← uniqueness

$$\sum a_i \vec{v_i} = \sum b_i \vec{v_i}$$

linear $a_i = b_i$

$$\sum \underbrace{(a_i - b_i)}_{=0} \vec{v_i} = 0$$

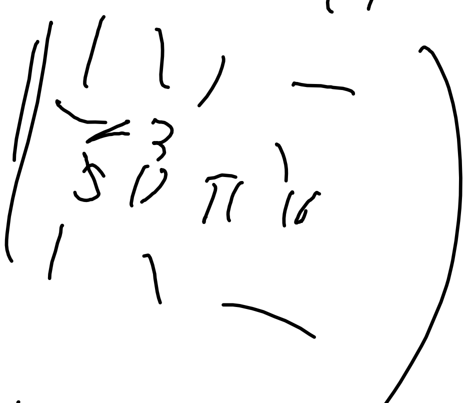


ker(T) , im(T)

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ker

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ im



dim V = k

$v_1, \dots, v_k$ span V $\Rightarrow$ lin indep

$v_1, \dots, v_n \in V$ lin indep $\Rightarrow$ basis

$\Leftrightarrow$

$T_{\vec{v}}(\vec{w}) = \vec{v} \cdot \vec{w}$

$: \mathbb{R}^3 \longrightarrow \mathbb{R}^1$

image: $\mathbb{R}$ 1 dimension!

$im(T) = \{\vec{0}\} \Leftrightarrow rk = 0 \Leftrightarrow T = 0$

$im(T) = \mathbb{R} \Leftrightarrow rk = 1 \Leftrightarrow T \neq 0$

$$\begin{array}{cc} & m \\ n & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array}$$

$\leq n$

$\leq m$ $\quad rk \leq \min\{n, m\}$

---

for which $\vec{v}$ is $T = 0$ transformation?

$\vec{v} = \vec{0} \Rightarrow T = 0$ transform

$\vec{v} \neq 0$. $T(\vec{w}) = 0$ for all $\vec{w}$?

$w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = v$ $\quad v \cdot w = 0$

$T(\vec{w}) = 0$ $\quad \vec{v} \cdot \vec{w} = 0$

$\vec{u} \perp \vec{v}$

$\vec{v} = 0$ $\quad im(T) = 0$

$ker(T) = \mathbb{R}^3$

$T(\vec{w}) = 0$ for all $\vec{w}$

$\underline{\vec{v} \neq 0}$ fixed $\vec{w}$ $T(\underline{\vec{w}}) \neq 0$

$ker(T) = \mathbb{R}^3$

e_1, e_2, e_3 is a b

$im(T)$ $\quad \& \; 0 \in K \; \cancel{\{0\}}$

$\begin{pmatrix} \vec{1} \\ \sqrt{2} \\ \pi, 5 \end{pmatrix}$

$\vec{0}$

$, \vec{v}$

$8\vec{v}$

$1.2\vec{v}$

Fix $\vec{v}$

$\underline{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$T(\vec{w}) = \vec{u} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{u}_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Then consider T

$\underline{\text{If } \vec{v} \neq 0}$ and $T = 0$

$T(\vec{w}) = 0$ for all $\vec{w}$

$\vec{v} \cdot \vec{w} = 0$ for all $\vec{w}$

$\vec{v}$ is perp to every vector

$T(\vec{v}) = \vec{v} \cdot \vec{v}$

$\text{im}(T)$ can only be $\{\vec{0}\}$

or $\mathbb{R}$

$\vec{v} \neq \vec{0}$, $\text{im}(T) \neq \{\vec{0}\}$

then $\text{im}(T) = \mathbb{R}$

$$\underline{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \cancel{\times} \begin{pmatrix} 1 \\ 9 \\ 9 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\vec{v} \neq \vec{0}$, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2 \neq 0$

$\quad\quad\ \ \|$

$\quad\ T(\vec{v})$

$\vec{v}$ can be anything

$\cancel{T}$ $\quad \vec{w}$ can anything

$T(\vec{w})$

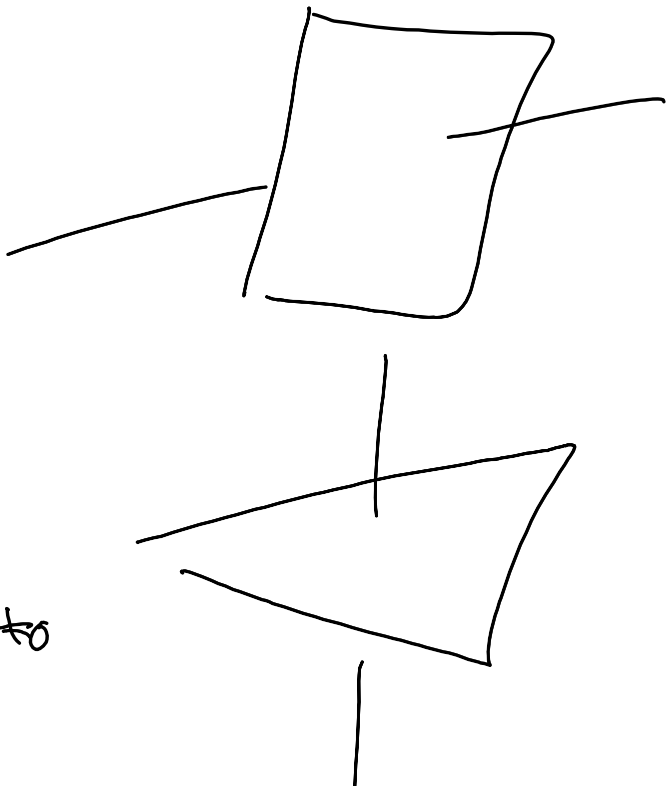
$\text{im}(T) = \mathbb{R}$

$\underline{\{1\}}$ $\quad c\,1 = c$

$\dim \ker T = 2$ $\dim \operatorname{im} T = 1$

$\neq \mathbb{R}^2$

$\dim \mathbb{R}^3 = 2$



$\vec{v} \neq 0$

$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}$

dim ker(T)

$3 = \underbrace{rk(T)}_{=1} + null(T)$

$\dim \ker(T) = null(T) = 2$

$\underline{\vec{v}} \rightsquigarrow \underline{\vec{x}, \vec{y}}$ in $\mathbb{R}$

i.e. $\vec{x}, \vec{y}$ are a basis for Ran (T)

$\vec{v} \rightsquigarrow$

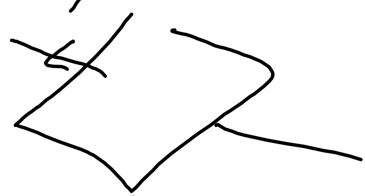
$\cancel{(\vec{x}, \vec{y}) = f(\vec{v})}$

$\vec{v}$

$c\vec{v}$

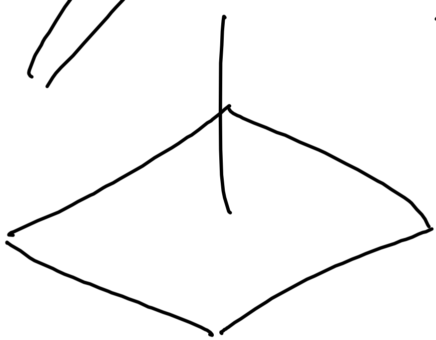
$\vec{v} + \begin{pmatrix}1\\0\\0\end{pmatrix}$

$\begin{pmatrix}1\\1\end{pmatrix}$



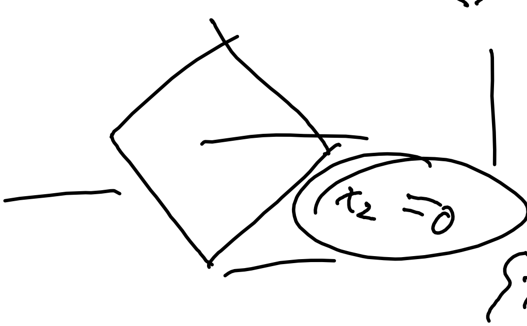
$\vec{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$

$v_1^2 - 2\sqrt{v_1 v_3}$



x_1, x_2, x_3

$x_1 = 0$



$x_2 = 0$

$x_1 = 0$

$\{x_1 + x_2 = 0\}$

$T(\vec{w}) = \vec{v} \cdot \vec{w}$

$\vec{v}$ fixed vector in $\mathbb{R}^3$

"describe" $\ker(T)$

$\vec{v}$

$\ker(T) \subseteq \mathbb{R}^3$

$\vec{w}$ s.t. $\vec{v} \cdot \vec{w} = 0$

$\vec{w}$ s.t. $\vec{v} \perp \vec{w}$

plane w/ normal vector $\vec{v}$
through $\vec{0}$

explicit example

$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\ker(T)$

$T(\vec{w}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

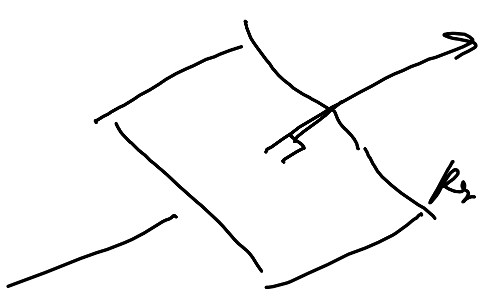
$= w_1 + 2w_2 + 3w_3$

$w_1 + 2w_2 + 3w_3 = 0$

plane

normal $= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

goes through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$



$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\ker(T) =$ set of all vectors perp to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}$

if $\vec{v} \neq 0$ then $\mathrm{im}(T) = \mathbb{R}$

formula $T(\vec{w}) = \vec{v} \cdot \vec{w}$

$\ker(T)$ set of solutions to $T(\vec{w}) = 0$.

subset of $\mathbb{R}^3$

2

3

$\mathbb{R}^{17}$

$\ker(T)$ subset of $\mathbb{R}^{17}$

16-dim subset of $\mathbb{R}^{17}$

$\dim \mathrm{im}\, T = 1$

rank nullity thm

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ linear

$$m = \underbrace{\dim \underbrace{\mathrm{im}\, T}_{\subseteq \mathbb{R}^n} + \dim \underbrace{\ker T}_{\subseteq \mathbb{R}^m}}_{\text{does depend } T}$$

indep of T

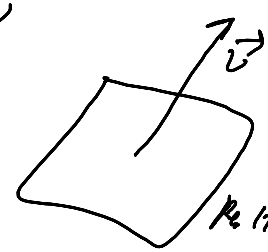
$\dim \ker T$

$= \dim$ inner span

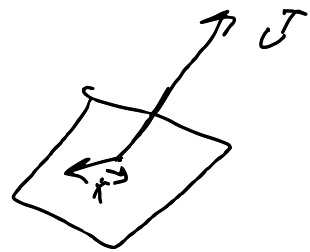
$= 1$

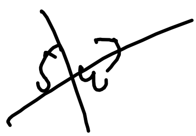
dim ker(T) = $\vec{w}$ s.t. $\vec{w} \perp \vec{v}$

2 dim thing



1. find $\vec{x} \neq 0$ $\vec{x} \perp \vec{v}$





$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$\underline{v_1} w_1 + \underline{v_2} w_2 + \underline{v_3} w_3 = 0$

$\vec{v}$

$\vec{x}$

2. goal: find $\underline{\vec{y}} \in \ker(T)$ s.t. $\{\vec{x}, \vec{y}\}$ lin. indep.

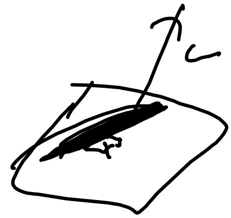
$\vec{y}$ exists because ker 2d

$\{\vec{x}, \vec{y}\}$ $\underline{\text{always}}$ lin dep, ~~for~~ all $\vec{y}$

pickon! $\vec{y} = c\vec{x}$ ~~for~~ all $\vec{y} \in \ker(T)$ plus

1. pick $\vec{x}$ in $Ra(T)$ nonzero

2. pick $\vec{y}$ in $Ra(T)$ not in span$(\vec{x})$



2.' $\vec{v}$? $\vec{x}$ $\to$ $\vec{y}$

$\vec{y} = \vec{v} \times \vec{x}$ cross product

$\vec{y} \perp \vec{v}$

$\vec{y} \in Ra(T)$

$\vec{y} \perp \vec{x}$ $\therefore$ $\vec{y}$ lin indep $\vec{x}$

$\vec{y} = \vec{0}$?!

$\|\vec{y}\| = \|\vec{v}\| \|\vec{x}\| \cancel{\sin\theta}$?

$\vec{v} \times \vec{v} \Rightarrow \vec{0}$

$\vec{v}, \vec{x}$ not parallel nonzero

$\Rightarrow \vec{v} \times \vec{x} \neq 0$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

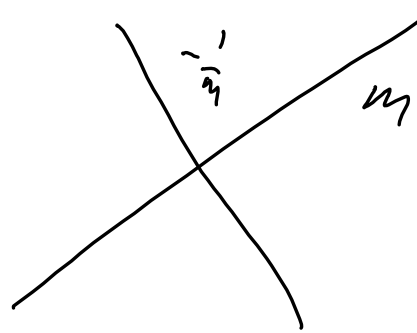
$$\left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right.$$

$$w_1 + 2w_2 + 3w_3 = 0$$

1 eq $\leftrightarrow$ rk 1

3 unknowns

2 choices $\leftrightarrow$ kern



$$w_3 = 0$$

$$w_1 + 2w_2 = 0$$

arbitrary

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$(w_2 = 0 \Rightarrow w_1 = 0)$

$$w_2 = 1$$

$$w_1 + 2 = 0$$

$$w_1 = -2$$

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \perp \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad v_1 \neq 0$$

$$\perp$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$u_3 = 0$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} = 0$$

if $v_1 \neq 0$ $\quad \begin{pmatrix} v_2 \\ -v_1 \\ 0 \end{pmatrix}$

if instead $v_2 \neq 0$ $\qquad u_1 = 0$

$\mathbb{R}^? \xrightarrow{T} \underline{\mathbb{R}^1}$ → cases based on dimension
0 is often special

0. count, rank nullity!

1. 1 small, so we can do cases

$\underline{rk = 0} \Longleftrightarrow \underline{\vec{v} = 0}$ ⇒ done

$\underline{rk = 1}$ $\quad \underline{\vec{v} \neq 0}$

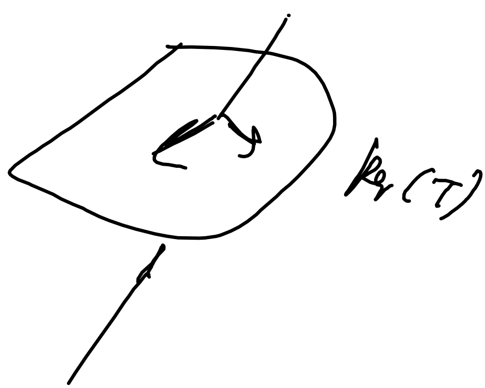
2. Consider $\vec{v} \neq 0$

$rk = 1$

basis im is $\{1\}$

ker?

geometrically, it's things perp to $\vec{v}$



2 dim

only need to find 2 vectors

if I find 1, then only the other remain

what I missed

explicit! $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \longrightarrow$ you can write explicit basis vectors for the plane

try to generalize

$\downarrow$

get stuck, split into cases

$$T\left(\underline{v_1}, \underline{v_2}, \underline{v_3}\right)$$

v^t

$$\{\mathbb{R}^3 \longrightarrow \mathbb{R}\} \longleftrightarrow 1\times 3 \text{ matrix}$$

$$\vec{v} \longleftrightarrow T$$

$\longrightarrow 3\times 1$ matrix

$\mathbb{R}^3$

$A \quad \begin{pmatrix} | & | & | & | & 0 & | \\ | & | & | & | & \vdots & | \\ | & | & | & | & 0 & | \end{pmatrix}$

A redundant?

short (in~~correct~~) yes

1. what is the def?

If v_i is a lin. combo. of the other cols for some i — v_i is in the space spanned by the others

Example $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$\vec{v}_1, \quad \vec{v}_2 = 2\vec{v}_1$

$\text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \cancel{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}\right)$ → useless redundant

Example not redundancy

$\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 5 \end{pmatrix} \neq c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

redundant $\vec{v}_1 + \vec{v}_2 = \vec{v}_3$

$\vec{v}_3 \in$ plane spanned $\vec{v}_1, \vec{v}_2$

matrices subspace

$A \longrightarrow \operatorname{im}(A)$

A $n \times m$ $\quad n \begin{pmatrix} 1 & & 1 \\ & & 1 \\ & & 1 \end{pmatrix}^{m}$

$\operatorname{rk}(A) \leq n$

$\leq m$

$\operatorname{rk} A \leq \min\{n, m\}$

Ex. find A s.t.

$\operatorname{im}(A)$ plane normal to $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ 4411 ≥ m, n

$\operatorname{im}(A)$ subset of $\mathbb{R}^3$

dim plane $= 2$

$\mathbb{R}^n \longrightarrow \mathbb{R}^3$

A $3 \times n$ matrix

3 rows

1 col. ?

dim $\operatorname{im}(A) = 2$

1 col, $\operatorname{rk}(A) \leq 1$

1 col

3 2 cols

3×3

$$\begin{pmatrix} | & | \\ | & | \\ | & | \\ T(e_1) & T(e_2) \end{pmatrix}$$

3×1000000000000

theoretically we can find A 3×3

im(A) plane normal to $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$$V = \{x + 3y + 2z = 0\}$$

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad \text{linear}$$

$$\text{im}(T) = V$$

- $\vec{e}_1 \longmapsto$?
- $\vec{e}_2 \longmapsto$??

$$\vec{e}_1 \overset{?}{\longmapsto} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$ in im(T) but not in V!

is $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ in V?

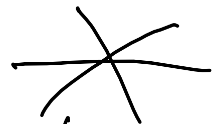
V is those perp $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

no!

$$\begin{bmatrix} \vec{e}_1 \longmapsto \vec{x} \\ \vec{e}_2 \longmapsto \vec{y} \end{bmatrix} = \begin{pmatrix} | & | \\ \vec{x} & \vec{y} \\ | & | \end{pmatrix}$$

[finding T is equivalent to finding $\vec{x}$ and $\vec{y}$]

$$\vec{e}_1 \longmapsto \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

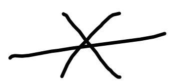


bad

$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ not in V

$\mathrm{im}(T)$ contains $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ not in V

$\mathrm{im}(T) \neq V$



$\vec{x}$, $\vec{y}$ in V,

$x_1 + 3x_2 + 2x_3 = 0$

$y_1 + 3y_2 + 2y_3 = 0$

$\vec{x} = \vec{y} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in V

$\mathrm{im}(T)$ is a subset (subspace) of V

T: $T(\vec{e}_1) = \vec{0}$ $T(\vec{e}_2) = \vec{0}$ $T(\vec{v}) = \vec{0}$

$$T \leftrightarrow \begin{array}{c} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \vec{x} \;\; \vec{y} \end{array} \quad \mathrm{im}(T) = ? \; \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$$

im(A) in terms of the columns?

span $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$

the image of a matrix is the span of its columns

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\text{im}(A) = \text{Span}(\vec{0}, \vec{0}) = \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$

$\vec{x}, \vec{y}$ in V ← our plane normal to $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$\text{im}(T)$ contained in V

$\quad\|$

$\text{span}(\vec{x}, \vec{y})$ contained in V

how do pick $\vec{x}, \vec{y}$ so that $\text{Span}(\vec{x}, \vec{y}) = V$?

~~ex~~) Certainly, $\vec{x}, \vec{y}$ must be in V, which is tested by the plane equation

Let V be the plane normal to $\begin{pmatrix}1\\3\\2\end{pmatrix}$.

Find $\vec{x}$, $\vec{y}$ s.t. span $(\vec{x}, \vec{y}) = V$.

$$\left.\begin{aligned} \vec{x} \cdot \begin{pmatrix}1\\3\\2\end{pmatrix} = 0 \\ \vec{y} \cdot \begin{pmatrix}1\\3\\2\end{pmatrix} = 0 \end{aligned}\right] \text{ necessary.}$$

mean that $\vec{x}$, $\vec{y}$ are in V

$\vec{0}$ in V

can we find $\vec{x} \perp \begin{pmatrix}1\\3\\2\end{pmatrix}$ non zero?

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \cdot \begin{pmatrix}1\\3\\2\end{pmatrix} = 0$$

$$x_1 + 3x_2 + 2x_3 = 0$$

1 eq,

3 unknown

$\underline{x_3 = 0}$. $\quad x_1 + 3x_2 = 0$

$x_1 = 0 \qquad x_1 = -3x_2$

$3x_2 = 0$

$x_2 = 0$

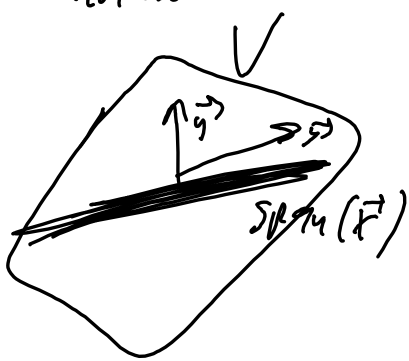
$$x_2 = 0 \rightarrow \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\cancel{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}} = \begin{pmatrix}-3x_2\\x_2\\0\end{pmatrix}$$

$x_2 = 1$

$$\vec{x} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}}_{\substack{\text{in } V \\ \text{nonzero}}} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{matrix} -3\cdot 1 \\ +1\cdot 3 \\ +0\cdot 2 \end{matrix} = -3+3 = 0$$



want to find $\vec{y}$ so that span $(\vec{x}, \vec{y}) = V$

another vector in V

$$\vec{y} = 2\vec{x}$$

$\vec{y}$ perp to $\vec{x}$ and in V

$\vec{x}$, $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

lin indep from $\vec{x}$

want $\boxed{\vec{y} \in V}$

which is indep. of $\vec{x} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$

$\vec{y} = c$ not indep of $\vec{x}$

$\vec{y} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ not in V

$\vec{y} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ ✓

$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$y_1 + 3y_2 + 2y_3 = 0$ (y lies in V)

$y_1 = 0$, $3y_2 + 2y_3 = 0$

$y_3 = -\frac{3}{2} y_2$

$y_2 = 1$

$y_3 = -3/2$

$\vec{y} = \begin{pmatrix} 0 \\ 1 \\ -3/2 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \\ -3/2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 0\cdot 1 + 1\cdot 3 + (-3/2)\cdot 2 = 0+3-3 = 0$$

$\vec{y}$ $\begin{pmatrix} 0 \\ 1 \\ -3/2 \end{pmatrix}$ in V

$$\vec{x} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ -3/2 \end{pmatrix}$$

Both in V ✓

lin indep? They're not multiples of each

$$c\vec{x} = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \neq \vec{y}$$

✓

$\operatorname{Span}(\vec{x}, \vec{y}) = V$??

both in V → $\operatorname{Span}(\vec{x}, \vec{y})$
lin indep is a 2d
subspace of V
$\dim V = 2$
∴ equal

$$A = \begin{pmatrix} -3 & 0 \\ 1 & 1 \\ 0 & -3/2 \end{pmatrix}$$

im = span cols,

$$\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix} \right\}$$

rank
span

$$\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$