"The Invertible Matrix Theorem"

$A$ $n \times n$ matrix

(*) 1. $A$ invertible matrix, $AA^{-1} = I_n$

$A^{-1}A = I_n$ $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$\Updownarrow$ ?

2. $\det(A) \neq 0$

3. $rk(A) = n$, $im(A) = \mathbb{R}^n$

solutions to $A\vec{x} = \vec{0}$

4. $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$, $ker(A) = \{\vec{0}\}$

5. columns are linearly independent

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\det = 1 \cdot 4 - 2 \cdot 2 = 0$

6. $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for any $\vec{b}$

(*) 7. $\boxed{rref(A) = I_n}$

$\text{im}(A) = \mathbb{R}^n$

represented by an $n \times m$ matrix

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ linear transformation

$\text{im}(T)$ = set of all vectors in $\mathbb{R}^n$ of the form $T(\vec{x})$ for some $\vec{x}$ in $\mathbb{R}^m$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\text{im}(T) = x\text{-axis}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ not in $\text{im}(T)$

$\text{im}(A) = \text{im}(T)$, where $T(\vec{x}) = A\vec{x}$

$T \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\text{im}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = x\text{-axis}$

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$\text{im}(A) = \mathbb{R}^n$

for all $\vec{b}$ in $\mathbb{R}^n$, there is some $\vec{x}$ in $\mathbb{R}^m$

so that $T(\vec{x}) = \vec{b}$

$A\vec{x} = \vec{b}$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$A$ invertible.

1. $A^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$T(\vec{x}) = A\vec{x}$

$T(\vec{e_1}) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2\vec{e_2}$

$T(\vec{e_2}) = \vec{e_3}$

$T(\vec{e_3}) = \vec{e_1}$

$T^{-1}(T(\vec{x})) = \vec{x}$

$\vec{x} = \vec{e_3}$

$T^{-1}(T(\vec{e_3})) = \vec{e_3}$

$T^{-1}(\vec{e_1}) = \vec{e_3}$

$T^{-1}(\vec{e_2}) = 1/2\ \vec{e_1}$

$T^{-1}(\vec{e_3}) = \vec{e_2}$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{ccc} T^{-1}\left(T(\vec{e}_1)\right) & = & T^{-1}\left(2\vec{e}_2\right) \\ \| & & \| \\ \vec{e}_1 & & 2T^{-1}(\vec{e}_2) \end{array}$$

$$\frac{1}{2}\vec{e}_1 = T^{-1}(\vec{e}_2)$$

$A\vec{x} = \vec{0}$ only solution is $\vec{x} = \vec{0}$

$A^{-1}(A\vec{x}) = A^{-1}(\vec{0})$

$\vec{x} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$A\vec{x} \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\|$

then $A\vec{x} \begin{pmatrix} z \\ 2x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leftarrow \vec{0}$

$z = 0$

$2x = 0$

$y = 0$

$x, y, z$ are all $0$

therefor $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$A\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$, $\ker(A) = \{\vec{0}\}$.

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$

find $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\left(\operatorname{im}(A) = \mathbb{R}^3\right)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} b/2 \\ c \\ a \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} z \\ 2x \\ y \end{pmatrix} \begin{matrix} \longleftrightarrow \\ = \\ \longleftrightarrow \\ \longleftrightarrow \end{matrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$A^{-1}A\begin{pmatrix} b/2 \\ c \\ a \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$z = a$

$2x = b$ $\quad x = \frac{b}{2}$

$y = c$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b/2 \\ c \\ a \end{pmatrix}$$

$$A \underbrace{\begin{pmatrix} b/2 \\ c \\ a \end{pmatrix}}_{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\underbrace{A^{-1} A}_{I_3 \,=\, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} b/2 \\ c \\ a \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b/2 \\ c \\ a \end{pmatrix}}_{= \begin{pmatrix} b/2 \\ c \\ a \end{pmatrix}} = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\Downarrow$

$$A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$\Downarrow$

first col

$$A^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underline{\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}}$$

$$A^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$$A^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftarrow ✓$$

$$A^{101} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - X$$

transition matrix?

$$\begin{pmatrix} 0.3 \\ 0.2 \\ 0.5 \end{pmatrix}$$

$n \times n$ matrix

positive: $A_{ij} > 0$ for all $i, j$

1) $A_{ij} \geq 0$ for all $i, j$

2) sum of entries in a column is 1

$$\sum_{i=1}^{n} A_{ij} = 1 \quad \text{for all } j$$


$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ $v_i$ = n° of people on website $i$

$AV$ distribution after one event of clicking

1. 100 big $A^2, A^3$

$A^2$ in terms of $A$'s entries

$A^{100} = B$

$(A^{100})_{ij} = ??$

$B_{ij} =$

$B$ pos, $A$ trans, $BA$ positive

$B_{ij} > 0$ for all $i, j$

$\begin{pmatrix} 0.? & 0.1 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 1/3 & 0 \\ 2/3 & 1 \end{pmatrix}$

$BA$ nonnegative

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X$$

$$BA = 0$$

wts $BA$ positive

$(BA)_{ij} \geq 0$ for all $ij$

$$(i^{th} \text{ row of } B) \cdot (j^{th} \text{ col of } A)$$

$$\|$$

$$(B_{i1}\ B_{i2} \cdots B_{in}) \cdot \begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{nj} \end{pmatrix}$$

$$\overset{\geq 0}{B_{i1}}\overset{\geq 0}{A_{1j}} + \overset{\geq 0}{B_{i2}}\overset{\geq 0}{A_{2j}} + \cdots + \overset{\geq 0}{B_{in}}\overset{\geq 0}{A_{nj}} \geq 0$$

$$\sum_{k=1}^{n} B_{ik}A_{kj} \overset{(?)}{=} 0$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \overset{\text{dot}}{=} \begin{pmatrix} & \bullet \\ & \end{pmatrix}$$

$(1,2)$ entry

$$1 \cdot 6 + 2 \cdot 8$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$\sum_{k=1}^{n} B_{ik} A_{kj} \geq 0$$

If $\downarrow = 0$, what happens?

$$B_{ik} A_{kj} \geq 0$$

so all $\underbrace{\underbrace{B_{ik}}_{>0} A_{kj} = 0}$ for $k = 1, 2, \dots, n$

$\downarrow$

$A_{kj} = 0$ for all $k = 1, \dots, n$

$A_{1j} = 0$

$A_{2j} = 0$

$A_{3j} = 0$

$\vdots$

$A_{nj} = 0$

$$0 = \sum_{k=1}^{n} A_{kj} = 1$$

$$0 = 1$$

$\underline{\text{if}}$ $\underbrace{(BA)_{ij} = 0}$ $\underline{\text{then}}$ $A_{kj} = 0$ for all $k$

false

$(BA)_{ij} \neq 0$. But we know $(BA)_{ij} \geq 0$. $\therefore (BA)_{ij} > 0$