$$\begin{vmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

"The Matrix Invertibility Theorem"

The following are equivalent, for A an $n \times n$ matrix.

1. A invertible.

?

(there is some $\underline{A^{-1}}$, $n \times n$ matrix, where

$AA^{-1} = I_n$, $A^{-1}A = I_n$)

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

2. The associate lin. transformation $T(\vec{x}) := A\vec{x}$ is invertible, $T^{-1} \circ T = \text{identity}$

3. $rk(A) = n$

(*) 4. $\det(A) \neq 0$

5. for every $\vec{b}$ in $\mathbb{R}^n$, there is a unique $\vec{x}$ in $\mathbb{R}$ so that $\underbrace{A\vec{x}}_{T(\vec{x})} = \vec{b}$

6. $\mathrm{rref}(A) = I_n$

7. $\mathrm{Ker}(A) = \{0\}$

8. $\mathrm{Im}(A) = \mathbb{R}^n$

$\simeq$

$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =$ origin

9. cols. are lin. indep!

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

7. $$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Claim: $$A^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A^{-1}A = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_n$$

2. $T(\vec{x}) = A\vec{x}$

$$T(\vec{e_1}) = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$= 2\vec{e_2}$$

$$T(\vec{e_2}) = \vec{e_3}$$

$$T(\vec{e_3}) = \vec{e_1}$$

$$T(T^{-1}(\vec{e_1})) = \vec{e_1}$$

$$\boxed{\begin{aligned} T(\vec{e_1}) &= 2\vec{e_2} \\ T(\vec{e_2}) &= \vec{e_3} \\ T(\vec{e_3}) &= \vec{e_1} \end{aligned}}$$

$$T^{-1}(\vec{e_1}) = ?$$

$$T(T^{-1}(\vec{e_1})) = T(?)$$

$$= \vec{e_1}$$

$$T(?) = \vec{e_1}$$

$$\vec{e_3}$$

$$T^{-1}(\vec{e_1}) = e_3$$

$$T^{-1}(\vec{e_3}) = \vec{e_2}$$

$$T^{-1}(2\vec{e_2}) = \vec{e_1}$$

$$T^{-1}(\vec{e_2}) = \frac{1}{2}\vec{e_1}$$

$$A^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

5. $A\vec{x}=\vec{b}$ has a unique $\vec{x}$ for all $\vec{b}$

If $\vec{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

want to find $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} z \\ 2x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$z = a$

$x = b/2$

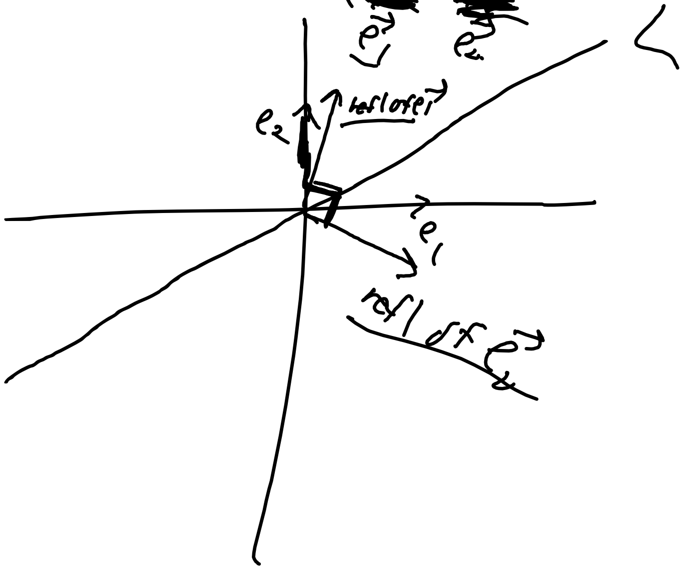
$y = c$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b/2 \\ c \\ a \end{pmatrix}$$

Reflection $A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ $\sqrt{a^2+b^2} = \sqrt{1}$

$\vec{e_1}$ $\vec{e_2}$

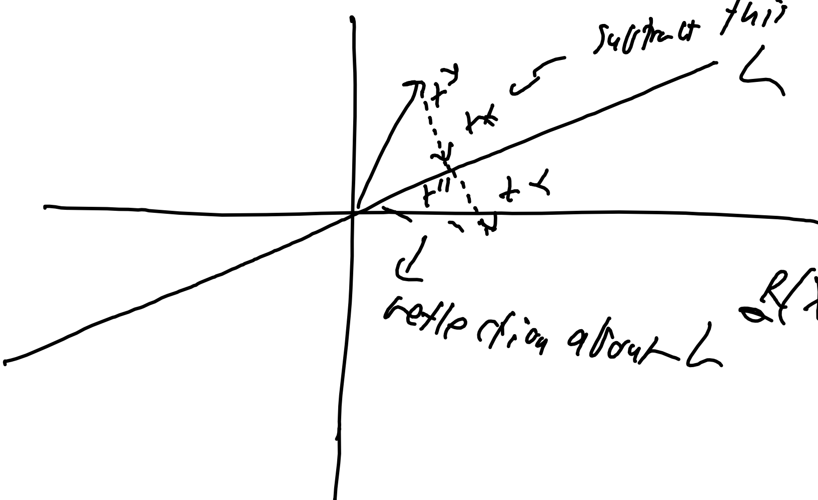
$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = 1$



$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} b \\ -a \end{pmatrix} = ab - ba = 0$

$\begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} b \\ -a \end{pmatrix}$

subtract this to get projection



reflection about L

$R(\vec{x}) = \vec{x} - 2x^\perp$

$-2\operatorname{proj}_L(\vec{x})$

1. A square matrix w/ 2 equal cols, then it's <u>not</u> invertible

<u>Systems of equations</u>, row reduction, rref

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \leftarrow \underline{\text{rank} = 1}$$

$\text{rank}(A) < n$

$\text{im}(A)$

$A\vec{x} = \vec{b}$ no solutions

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 2 \\ 3 & 6 & 3 \end{pmatrix}$$

$A\vec{x} = \vec{b}$, given $\vec{b}$, how many solutions $\vec{x}$ are there?

$A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$? How many?

2. reflection

rotation $\longleftrightarrow$ rotate back

scaling

projection

think geometrically!

3. 4. permutation matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A\vec{e_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{e_3}$$

$$A\vec{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \vec{e_2}$$

$$A\vec{e_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{e_1}$$

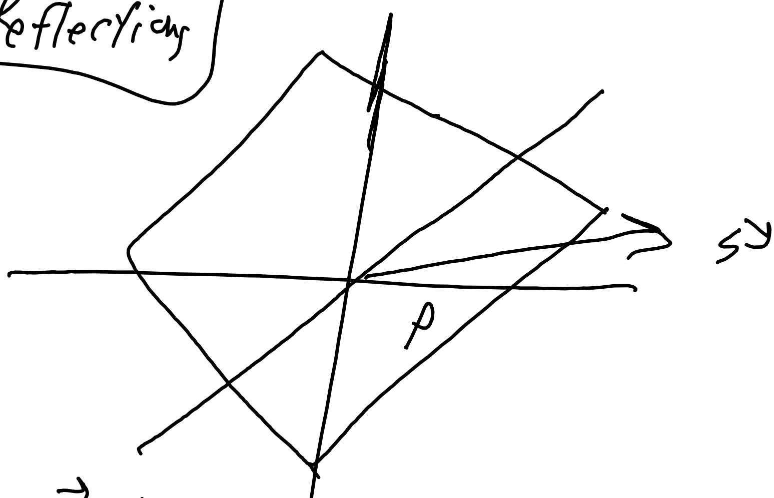
$\vec{e_1}$ $\vec{e_2}$ $\vec{e_3}$

$\vec{e_3}$ $\vec{e_2}$ $\vec{e_1}$

5. A, B inv

$T(\vec{x}) = \underbrace{A}_{\text{shoes}}(\underbrace{B}_{\text{socks}}\vec{x})$ invertible?

Reflections



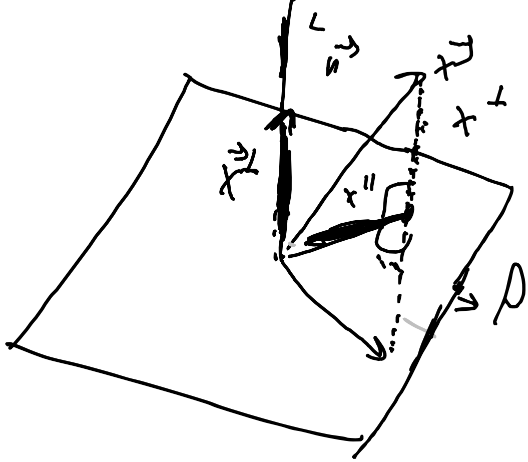
$\vec{n} \perp \vec{v}$ for all $\vec{v}$ on P ← plane

(?)

$\mathrm{Refl}(\vec{x}) = \vec{x} - 2x^{\perp}$

$= \vec{x} - (\vec{x} \cdot \vec{u})\vec{u}$

(?)



$L \perp P$

$Refl(x) = \vec{x} - 2\vec{x}^{\perp}$

$\vec{x}^{\parallel}$ proj of $\vec{x}$ onto plane

$\vec{x}^{\perp}$ proj of $\vec{x}$ onto L

$\vec{x}\cdot\vec{y}$
$=\|x\|\|y\|\cos\theta$

$\vec{n}$ in L unit length

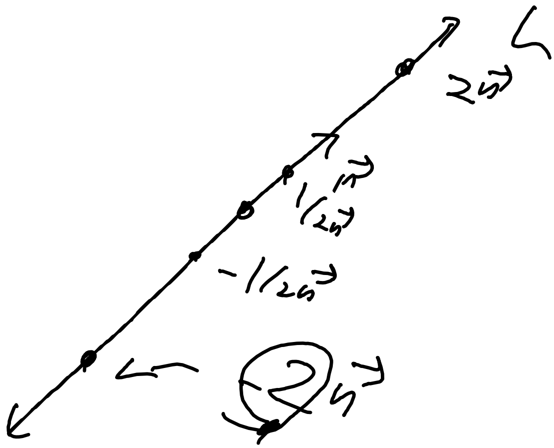
magnitude direction

$\|\vec{n}\|=1$

$(\vec{x}\cdot\vec{n})\,\vec{n} = \vec{x}^{\perp}$, proj onto L

$= \vec{x} - 2(\vec{x}\cdot\vec{n})\,\vec{n}$

$\vec{x} \cdot \vec{n}$ <u>scalar</u>



T/F A transition matrix, A^{100} pos $\Rightarrow$ A^{101} pos ?

$\Updownarrow$

every column is "distr vector", entries ≥ 0, sum along each column is 1

$$\begin{array}{c} \geq 0 \\ \geq 0 \\ \geq 0 \end{array} \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

$\Downarrow$

$a_{ij} \geq 0$ for all i, j

$\sum_{i=1}^{n} a_{ij} = 1$ for all j

jth column

$\Updownarrow$

positive if all entries > 0

regular if $A^n \to$ positive

"eventually positive"

$A^{101} = A^{100} A$

if A positive, done

$B = A^{100}$ positive

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^{100} = ?$

$AB \overset{(?)}{=}$ pos

trans, pos

$B_{ij} > 0$ for all i, j

$\neq$ transition

(hypotheses

A trans. matrix

B trans. matrix s.t. $B_{ij} > 0$ for all i, j

$(B = A^{100})$

(BA positive)

claim: AB positive

$(AB)_{ij} > 0$ for all i, j

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & - \\ - & - \end{pmatrix}$$

$=$

i^{th} row of A $\cdot$ j^{th} column of B

$$(A_{21}B_{14} + A_{22}B_{24} + A_{23}B_{34} + A_{24}B_{44} + \cdots + A_{2n}B_{n4})$$

$(AB)_{11}$ if $n=2$

$n=3$

$(AB)_{11}$

$\leftarrow$ $(2,4)$ entry

$(AB)_{24}$

$A_{21}B_{14} + A_{22}B_{24} + \cdots + A_{2n}B_{n4}$

$$\begin{pmatrix} A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \end{pmatrix} \begin{pmatrix} B_{14} \\ B_{24} \\ \vdots \\ B_{n4} \end{pmatrix}$$

$$= \sum_{k=1}^{n} \underbrace{A_{ik}}_{\geq 0} \underbrace{B_{kj}}_{> 0} \geq 0$$

A^{100}

A $n \times m$

$A \cdot A$?

$(n \times m) \cdot (m \times l) = n \times l$

cols row

$\mathbb{R}^m \to \mathbb{R}^n$ $\mathbb{R}^l \to \mathbb{R}^m$ $\mathbb{R}^l \to \mathbb{R}^n$

$(n \times m) \cdot (n \times m)$

$n = m$ necessary

A square, $n \times n$

A^{100} also $n \times n$

$$(AB)_{ij} = \sum_{k=1}^{n} \underbrace{\underbrace{A_{ik}}_{\geq 0}\ \underbrace{B_{kj}}_{\geq 0}}_{\geq 0}$$

≥ 0

$= 0$? $\quad \underbrace{A_{ik}}\ \underbrace{B_{kj}}_{>0} = 0$ for all k

$A_{ik} = 0$ for all k

If $= 0$

Then $A_{ik} = 0$ for all k

i^{th} row of A is all 0s

$$\underbrace{\begin{pmatrix} 1/2 & 1/2 & 1/3 \\ 0 & - & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}}_{A} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} - & \cdot & \cdot \\ 0 & 0 & 0 \\ - & \cdot & \cdot \end{pmatrix}$$

AC also has a row of 0s

Also row of zeros, but not!

$$\begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0 & - & 0 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}$$

$AB \to 0$'s

rows of A

$\|$

$BA \to$ cols

rw

$B = A^{100}$

$$\sum_k \underbrace{B_{ik}}_{\geq 0} \underbrace{A_{kj}}_{\geq 0} = 0 \implies A_{kj} = 0 \text{ for all } k$$

a column of A must be 0

$$\operatorname{im}(A) = \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

$A_{m \times n}$ matrix

in $\mathbb{R}^m$ $T(\vec{x})$

$$T(\vec{x}) = A\vec{x}$$

$$\begin{array}{c} \\ m \\ \\ \end{array} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} \\ \vec{x} \\ \\ \end{pmatrix} n = \begin{pmatrix} \\ A\vec{x} \\ \\ \end{pmatrix} m$$

$$\begin{array}{ccc} n & 1 & 1 \end{array}$$

The set of all $\vec{b} \in \mathbb{R}^n$

so that there is some $\vec{x}$

w/ $A\vec{x} = \vec{b}$

$y = f(x)$

for x?

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = x^2$$

$\operatorname{im}(f)$ $\qquad$ 4 (?)

$\quad\shortparallel$

$[0, \infty)$ $\qquad$ $f(2) = 4$

$\therefore$ $4 \in \operatorname{im}(f)$

-1 not in $\operatorname{im}(f)$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \underset{\text{output}}{2} \times \underset{\text{input}}{2}$ matrix

$T(\vec{x}) = A\vec{x}$ $\qquad f(x) = x^2$

$T: \underset{\text{input}}{\mathbb{R}^2} \longrightarrow \underset{\text{output}}{\mathbb{R}^2}$ $\qquad M_{2\times 2}(\mathbb{R}) \xrightarrow{f(x)=x^2} M_{2\times 2}\mathbb{R}$

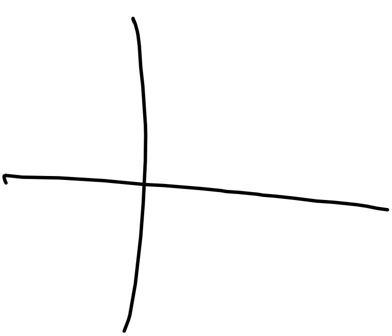
$\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ $\qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_2 \longrightarrow e_1$ $\quad c^1$

$\mathbb{Q}^2 \longrightarrow \mathbb{Q}^2$

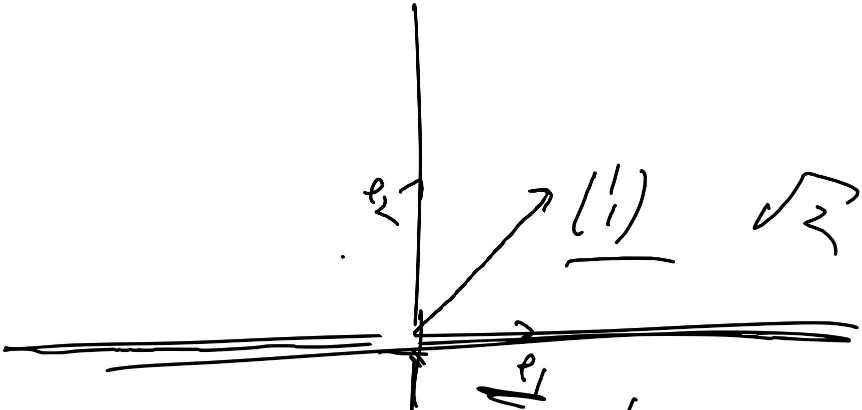
$* \; \mathbb{Z}^n$

$A\vec{e}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A\vec{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$



$\mathbb{C}^2 = ?$



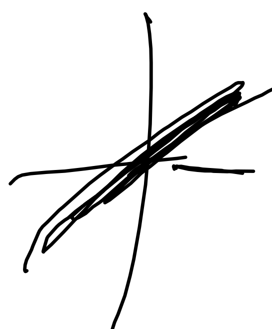
$\begin{pmatrix}1\\-1\end{pmatrix} \mapsto 0$

$\begin{pmatrix} 1/\sqrt{2} & - \\ 1/2 & \end{pmatrix}$

orthog. proj onto $\begin{pmatrix}1\\1\end{pmatrix}$

scale by $\sqrt{2}$

$\operatorname{im}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \{y=x\}$



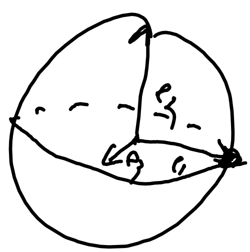
$\left\{ \begin{pmatrix} c \\ c \end{pmatrix} \,\middle|\, c \in \mathbb{R} \right\}$

$$\begin{pmatrix} c \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & d \end{pmatrix}$$

im(A) things that look like $A\vec{x}$

$\vec{x}$ in $\mathbb{R}^3$



(?) im(A) = span $\left\{ \begin{array}{cc} x & y \\ \begin{pmatrix} 1 \\ a \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array} \right\}$ = set of all lin combos of $\begin{pmatrix} 1 \\ a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

subset of $\mathbb{R}^3$

claim. $\begin{array}{c} z \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$ not in this span

$a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\0\end{pmatrix}$ a, b real numbers

$\mathrm{Span}_{\mathbb{R}}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ is the set of

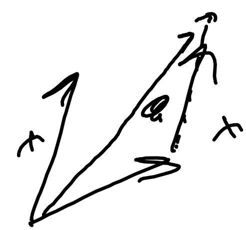
$\underbrace{c\begin{pmatrix}0\\0\\0\end{pmatrix}}_{0}$

span$_a$ span$_b$

$\begin{pmatrix}a\\0\\0\end{pmatrix} + \begin{pmatrix}0\\b\\0\end{pmatrix}$

$a=2$ $\begin{pmatrix}a\\8\\0\end{pmatrix}$

$= \begin{pmatrix}a\\b\\0\end{pmatrix}$



precisely the vectors like this are the span of $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right.$

$\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\begin{pmatrix}1\\3\\0\end{pmatrix}$, $\begin{pmatrix}-1\\-5\\0\end{pmatrix}$, $\begin{pmatrix}71\\17\\0\end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ eigenvalues $i, -i$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\mathbb{C}^2 \longrightarrow \mathbb{C}^3$

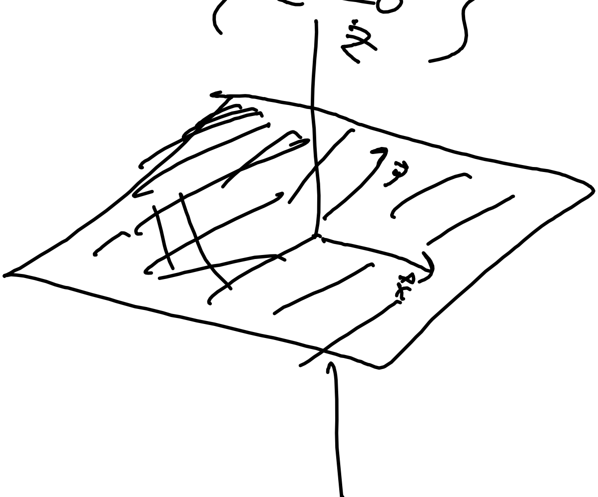
im(A) ? span$\{$cols of $A\}$

$= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$

$= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \ a\ b \text{ reals} \right\}$

$\{ z = 0 \}$



$= \text{im}(A)$

claim: $\text{im}(A) = \text{span}\{\text{cols of } A\}$

$\exists$ generic

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}$$

$\exists$ generic

$$\cancel{x}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}$$

A $n \times m$ $\quad A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix}$ $\quad \vec{v}_i$ i^{th} col of A

$A\vec{x}$

$$\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$\sum_{i=1}^{m} a_i \vec{v}_i = a_1\vec{v}_1 + \cdots + a_n \vec{v}_m$$

$$A\begin{pmatrix}x_1\\ \vdots \\ x_m\end{pmatrix}$$

$$\|$$

$$\begin{pmatrix}x_1\\ \vdots \\ x_m\end{pmatrix} = x_1\vec{e}_1+\dots+x_m\vec{e}_m$$

$$x_1 \underbrace{A\vec{e}_1}_{\vec{v}_1} + \dots + x_m \underbrace{A\vec{e}_m}_{\vec{v}_m}$$

$$\|$$

$$x_1\vec{v}_1+\dots+x_m\vec{v}_m$$

$$A \;\|$$

$$\underbrace{\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix}}_{\text{cols of } A}\underbrace{\begin{pmatrix}x_1\\ \vdots \\ x_m\end{pmatrix}}_{\text{coefficients of lin combo}} = \underline{x_1}\,\vec{v}_1+\dots+\underline{x_m}\,\vec{v}_m$$

lin combo of cols

$$\text{im } A = \text{Span}\{\text{cols of } A\}$$

A^2 , A^3

$\parallel$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{im}(A^2) = \mathrm{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

x-axis

$$A^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}$$

$\mathrm{im}(A) = \mathrm{im}(B)$

$\underline{A\vec{x} = 0}$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

rk / imag	2	1	0	
kernel's	1	2	3	nullity +
	3	3	3	$= \dim \mathbb{R}^3$

"rank nullity thm"

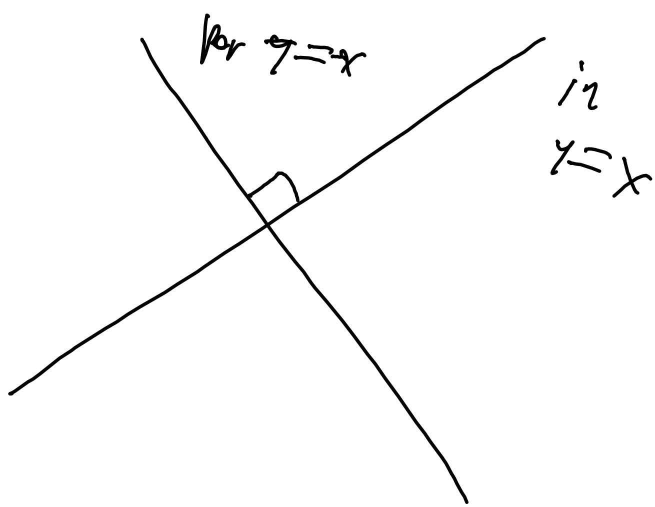
A $n \times m$ matrix

$T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$$\begin{array}{ccccc} \operatorname{rk} A & + & \operatorname{null} A & = & m \\ \| & & \| & & \\ \dim \operatorname{im} A & + & \dim \ker A & = & m \end{array}$$

"nilpotent"

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



Kernels

null space A $n \times m$

$\ker(A)$ is the set of $\vec{x}$ in $\mathbb{R}^m$ so that $A\vec{x} = \vec{0} \leftarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ in $\mathbb{R}^n$

- $\{\vec{x} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{0}\}$

- it's the solutions to equation $A\vec{x} = 0$

Is there $\vec{x}$ sol. $A\underline{\vec{x}} = \vec{b}$?

1. Is this solvable? im
   $\vec{b}$ in im A

   $\Downarrow$?

   $A\vec{x} = \vec{0}$ solvable

   $\vec{0}$ is in im A

$\vec{b} = \vec{0}$

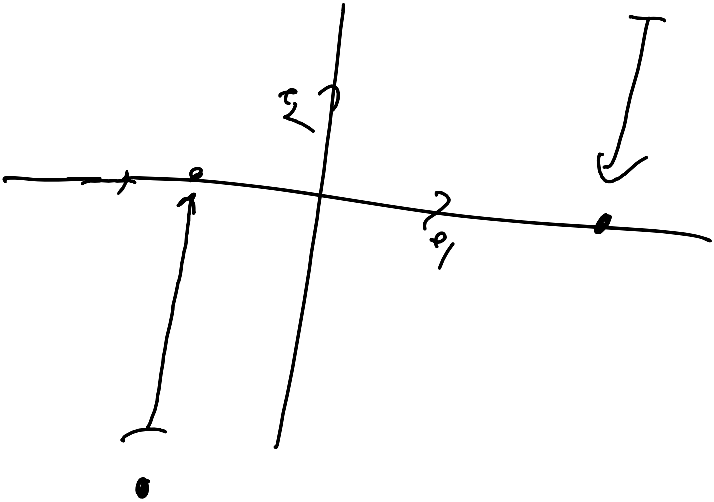
$\vec{x} = \vec{0}$

2. $\ker(A)$ is "how much imprecision"

how many

$$\boxed{A = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}, \quad T\begin{pmatrix}x\\ y\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$$

$$= \begin{pmatrix}x\\ 0\end{pmatrix}$$



<u>Ker(A)</u>

$\begin{pmatrix}x\\ y\end{pmatrix}$ s.t. $A\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

$x = 0$

y free

$$\left\{ \begin{pmatrix}0\\ y\end{pmatrix} \; y \in \mathbb{R} \right\}$$

$\underline{a}\vec{e}_1 + \underline{b}\vec{e}_2$ what $a = 0$

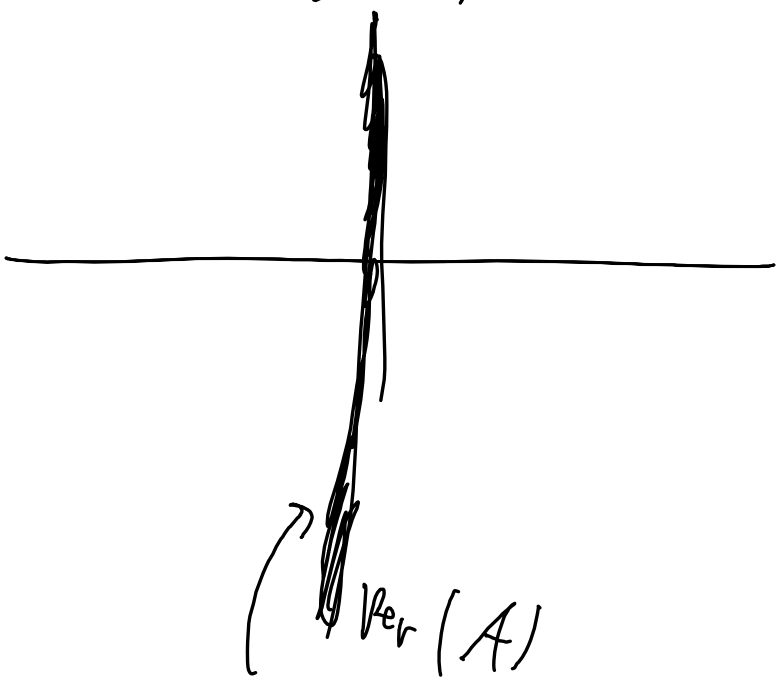
$\{ b\vec{e}_2 \}$ <u>(span $\vec{e}_2$)</u>

$\ker(A) = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \,\middle|\, y \text{ in } \mathbb{R} \right\}$.

$= \boxed{\operatorname{Span} \vec{e}_2}$.

= the set of all $\begin{pmatrix} 0 \\ y \end{pmatrix}$ when y ranges over all real numbers

= y-axis



$A\vec{x} = \vec{0}$

$\Rightarrow$

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;?\qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\vec{x} = A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$x = 1$

y is anything

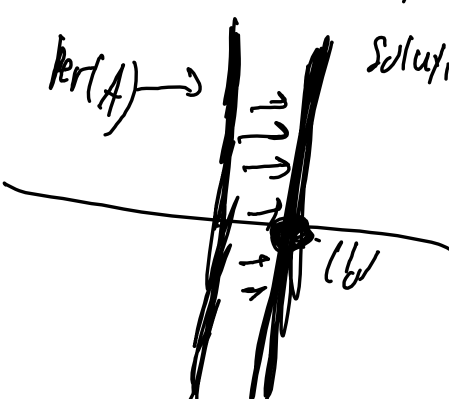
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \{x = 1\}$$

no cond. on y

Set of all $\begin{pmatrix} 1 \\ y \end{pmatrix}$ y in $\mathbb{R}$ anything



Solution to $A\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

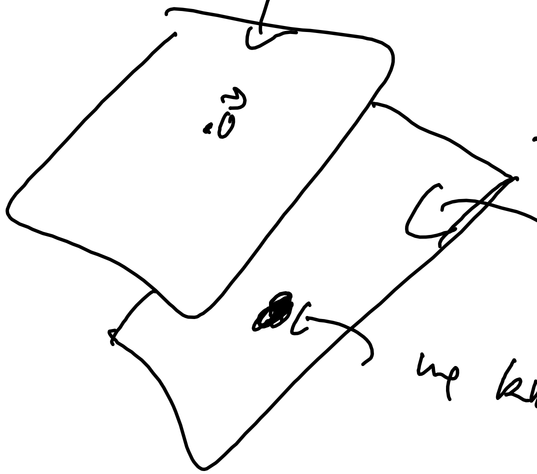
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$A\vec{x}=\vec{b}$

$f(\vec{x})=\vec{b}$

$f\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1^2\\x_2^2\cos(x_1)\end{pmatrix}$

ker(A)



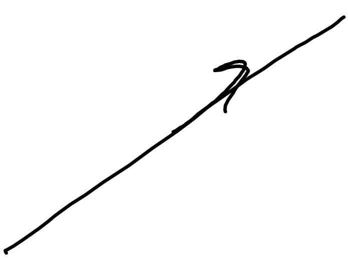
$A\vec{x}=\vec{b}$

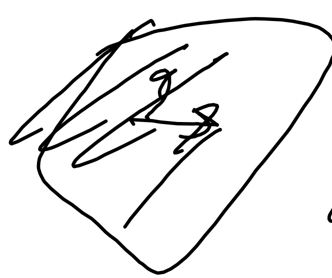
set of all solutions

we know this is a solution

$\begin{pmatrix}1\\2\\3\end{pmatrix}$

dim ker A $\in \mathbb{N}$





n natural number $n \geq 1$ $(n=0)$

$\cancel{a_0} + a_1 x + a_2 x^2$

$\mathbb{R}^n$ order n many lists of n real #s

$\begin{pmatrix}1\\2\end{pmatrix} \neq \begin{pmatrix}2\\1\end{pmatrix}$

we write this in a column

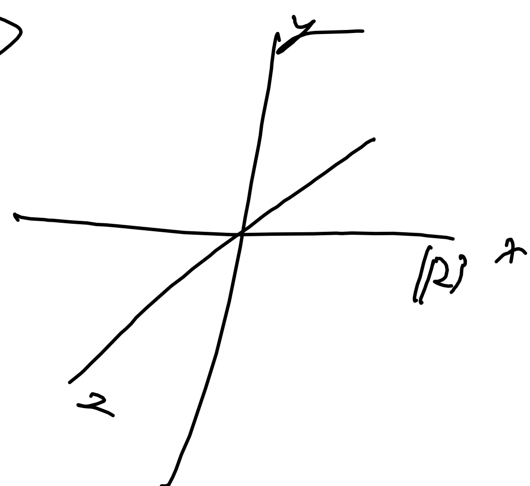
$\mathbb{R}^2$ (x, y) in $\mathbb{R}^2$

$\begin{pmatrix}x\\y\end{pmatrix}$

$\mathbb{R}^0$ •

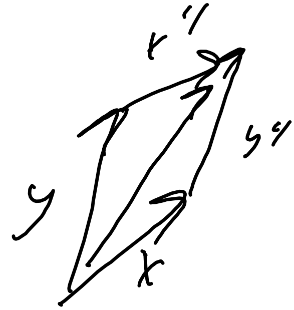
$\mathbb{R}^1$

$\mathbb{R}^3$



$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

and $a = c$
$b = d$.



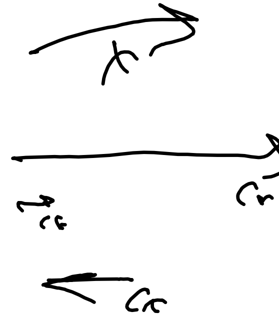
$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$a_i = b_i$ for all i

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \overset{(\text{def})}{=} \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

$\mathbb{R}^n, =, +, \cdot$

$$c \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \overset{(\text{def})}{=} \begin{pmatrix} ca_1 \\ \vdots \\ ca_n \end{pmatrix}$$



$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \overset{(\text{def})}{=} \sum a_i b_i$$