Least squares

$A\vec{x} \doteq \vec{b}$?

$\vec{x}^*$ closest

$||\vec{b} - A\vec{x}^*|| \leq ||\vec{b} - A\vec{x}||$ for all $\vec{x}$

$\vec{x}^*$ is a least squares sol to $A\vec{x} = \vec{b}$

$\Updownarrow$

$\vec{x}^*$ is a solution to $A^T A\vec{x} = A^T\vec{b}$

$A^T A$ not invertible ?

$A = 0$.

$A\vec{x}^* = \vec{0}$

$A\vec{x} = \vec{0}$ all $\vec{x}$

$\ker(A) = \{\vec{0}\}$

$\Updownarrow$

A^TA invertible

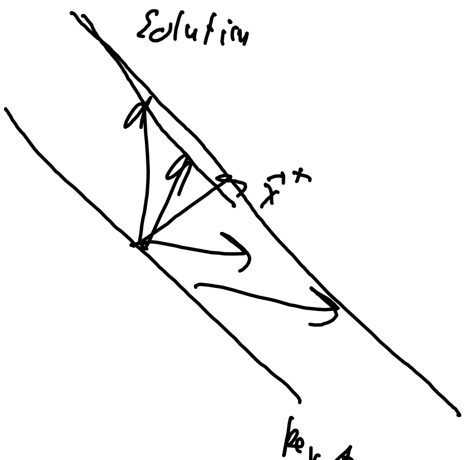
$\ker(A^TA) = \ker(A)$

$\vec{x}^*$ some least square solution

$\vec{y}$ in $\ker(A)$

$A(\vec{x}^* + \vec{y}) = A\vec{x}^* + A\vec{y} = A\vec{x}^*$

$\Rightarrow$ $\vec{x}^* + \vec{y}$ is also a least square solution



$$\boxed{\underline{A\vec{x}} = \underline{\vec{b}}}$$

A $n \times m$ matrix

$\vec{x}$ $m \times 1$ vector, in $\mathbb{R}^m$

$\vec{b}$ $n \times 1$, in $\mathbb{R}^n$

$\mathbb{R}^m \longrightarrow \mathbb{R}^n$

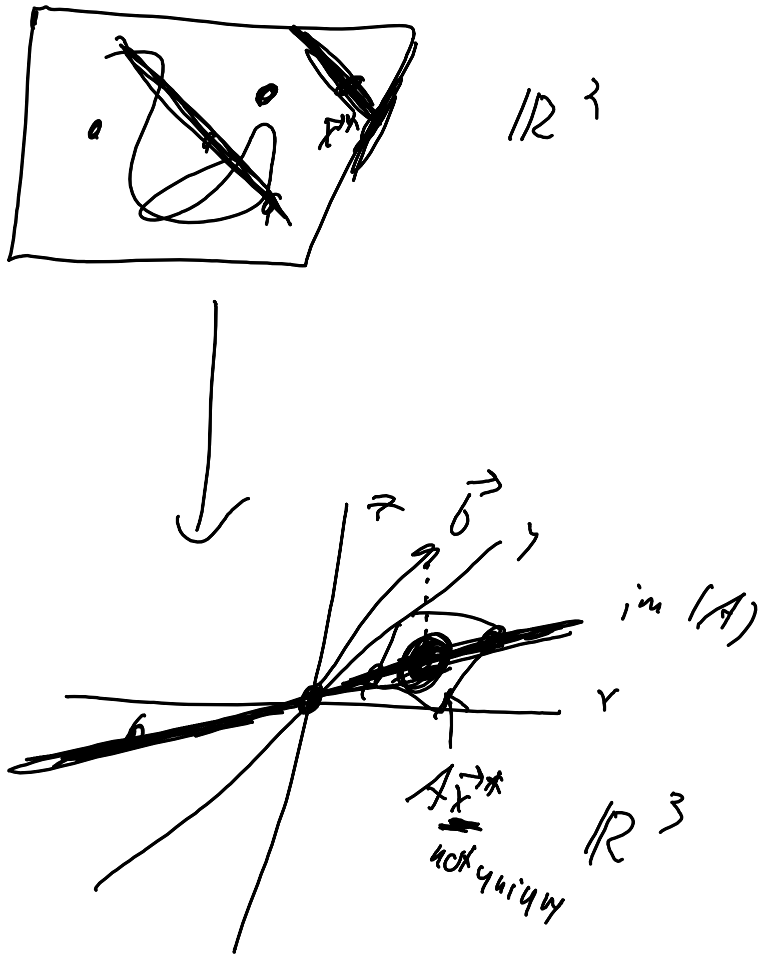
$n \neq m$

A 3×2

$\vec{x}$ $\mathbb{R}^3$ $\qquad \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\vec{b}$ $\mathbb{R}^3$

$A\vec{x}^*$ closest to $\vec{b}$ that we can get



$\vec{x}^*$
$\mathbb{R}^3$
z
$\vec{b}$
y
im (A)
x
$A\vec{x}^*$
$\mathbb{R}^3$

$\boxed{\text{1E T/F}}$

Are there real invertible 3x3 matrices A, S

$$S^{-1} A S = 2A$$

$$\begin{tikzcd}
\bullet \arrow[r, "S"] \arrow[d, dotted, "A"'] & \bullet \arrow[d, "A"] \\
\bullet & \bullet \arrow[l, "S^{-1}"]
\end{tikzcd}$$

$SA = AS \implies S^{-1}AS = A$

not always the case

S, A commute $\implies$ impossible

$\implies A = 2A$ $\Leftarrow$

$$\underline{SAS^{-1}} \overset{(?)}{=} \underline{2A}$$

det

move into the commutative world?

$ab = ba \quad a, b \text{ in } \mathbb{R}$

$$\left[\det(SAS^{-1}) = \det(2A)\right]$$

$A + B = C$

$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ ← commutative

$\det(\underline{A}B) = \underline{\det(A)}\ \underline{\det(B)}$

$= \det(S)\det(A)\det(S^{-1})$

$= \det(S)\det(A)\dfrac{1}{\det(S)}$

$= \det(S)\dfrac{1}{\det(S)}\det(A)$

$= \det(A)$

$\det(2A)$

$\|$

$2^3\det(A)$

$\det(A) = 2^7 \det(A)$

A inv $\Rightarrow \det(A) \neq 0$

$1 = 8$ absurd

$\Downarrow$

false

4. find orthogonal

$\begin{pmatrix} a & b & c \\ c & d & 1 \\ e & f & 0 \end{pmatrix}$ orthog

$b^2 + f^2 = 1$

$c = d = 0$

$\begin{pmatrix} b \\ 0 \\ f \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ e \end{pmatrix}$

$a^2 + c^2 + e^2 = 1$

$b^2 + d^2 + f^2 = 1$

$ab + cd + ef = 0$

$c = d = 0$

$b^2 + f^2 = 1 \quad a^2 + e^2 = 1$

$c = d = 0$

$ab + ef = 0$

2.

find abcdefg $\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \begin{pmatrix} b \\ i \\ g \end{pmatrix}, \begin{pmatrix} c \\ e \\ k \end{pmatrix}$

s.t. orthonormal

find $\boxed{\text{some}}$ a, b, c, d, e, f, g,

~~all~~

$\begin{pmatrix} b \\ i \\ g \end{pmatrix} = \begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix}$ $b = g = 0$

$e = d = 0$

rk · nullity

A $n \times n$ matrix $\ker(A) = 0 \Longleftrightarrow \operatorname{im}(A) = \mathbb{R}^n$

A ortho $\Rightarrow A^T$ ortho

cols of A ONB rows of A ONB

$$\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \begin{pmatrix} b \\ 1 \\ g \end{pmatrix}, \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}$$

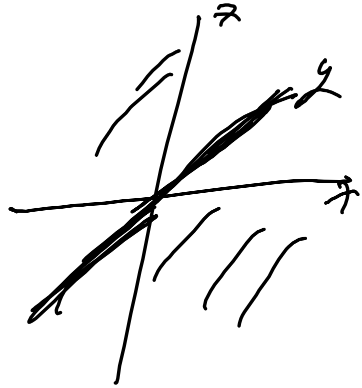
$$b^2 + 1^2 + g^2 = 1^2$$

$$b = g = 0$$

$$- \begin{pmatrix} a \\ d \\ f \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}$$

xz plan

xz plu



$$0 = \begin{pmatrix} a \\ d \\ f \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = d$$

$$d = e = 0$$

$$0 = \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e$$

$$\begin{pmatrix} a \\ 0 \\ f \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 0 \\ 1/2 \end{pmatrix}$$

~~Let f = 0 a = 1~~ $\cancel{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 0 \\ 1/2 \end{pmatrix}}$

$$\begin{pmatrix} a \\ 0 \\ f \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} c \\ 0 \\ 1/2 \end{pmatrix}$$

$$\underline{a^2+f^2=1} \qquad \underbrace{c^2+\left(\frac{1}{2}\right)^2=1}$$

$$\underline{ac+\frac{f}{2}=0}$$

$$\begin{pmatrix} a \\ c \end{pmatrix}, \quad \begin{pmatrix} -c \\ a \end{pmatrix}$$

$$c^2+\frac{1}{4}=1$$

$$c^2=\frac{3}{4}$$

$$c=\pm\frac{\sqrt{3}}{2}$$

choice: $c=\frac{\sqrt{3}}{2}$

$$\underline{\begin{pmatrix} a \\ 0 \\ f \end{pmatrix}}, \quad \underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \quad \underline{\begin{pmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \end{pmatrix}}, \qquad \begin{pmatrix} 1/2 \\ 0 \\ -\sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \end{pmatrix}$$

$$\underbrace{a^2+f^2=1}, \quad a\frac{\sqrt{3}}{2}+\frac{f}{2}=0, \quad \underbrace{a\sqrt{3}+f=0}_{f=-a\sqrt{3}}$$

$$a^2+(-a\sqrt{3})^2=1$$

$$a^2+3a^2=1, \quad 4a^2=1, \quad a^2=\frac{1}{4}, \quad a=\pm\frac{1}{2}, \quad a=\frac{1}{2} \rightsquigarrow f=-\frac{\sqrt{3}}{2}$$

6. A 4×4

rows $\vec{v}_i$

$\det(A) = 8$

$$\det\begin{bmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_2 + 7\vec{v}_4 & - \\ - & \vec{v}_3 - 2\vec{v}_4 & - \\ - & \vec{v}_4 & - \end{bmatrix} \qquad \begin{bmatrix} - v_1 \\ - v_2 \\ - v_3 \\ - v_4 \end{bmatrix}$$

multilinearity of det

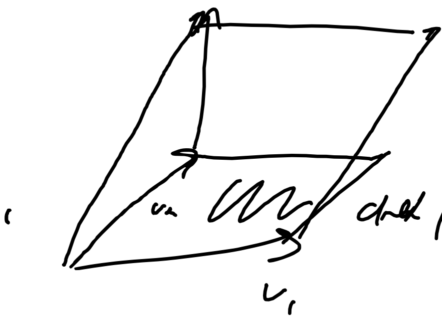
$$\det\begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \\ - & R_4 & - \end{bmatrix}$$

$$\underline{T(\vec{x})} = \begin{bmatrix} - & \vec{x} & - \\ - & R_2 & - \\ - & R_3 & - \\ - & R_4 & - \end{bmatrix} : \mathbb{R}^{1\times 4} \longrightarrow \mathbb{R}$$

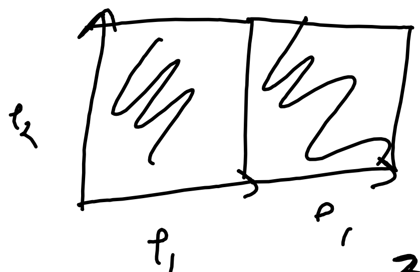
linear

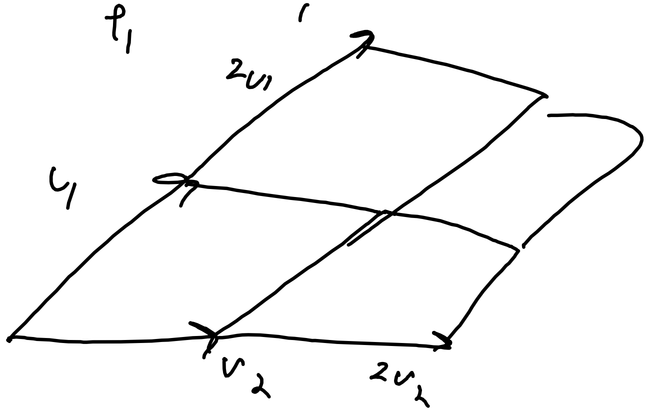
$$\det(A+B) \neq \det(A) + \det(B)$$

$$\begin{bmatrix} - v_1 - \\ - v_2 - \end{bmatrix} + \begin{bmatrix} - w_1 - \\ - w_2 - \end{bmatrix} = \begin{bmatrix} - v_1 + w_1 - \\ - v_2 + w_2 - \end{bmatrix}$$



$\det\begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix}$





$$\det\begin{pmatrix} — & v_1 & — \\ — & v_2 + 7v_4 & — \\ — & v_3 - 2v_4 & — \\ — & v_4 & — \end{pmatrix}$$

$$= \det\begin{pmatrix} v_1 \\ v_2 \\ v_3 - 2v_4 \\ v_4 \end{pmatrix} + \cancel{\det\begin{pmatrix} v_1 \\ 7v_4 \\ v_3 - 2v_4 \\ v_4 \end{pmatrix}} \to 0$$

$$= \det\begin{pmatrix} v_1 \\ v_2 \\ v_3 - 2v_4 \\ v_4 \end{pmatrix} + \underbrace{7\det\cancel{\begin{pmatrix} v_1 \\ v_4 \\ v_3 - 2v_4 \\ v_4 \end{pmatrix}}}_{=0} \to 0$$

$$= \underbrace{\det\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}}_{=8} - 2\det\cancel{\begin{pmatrix} v_1 \\ v_2 \\ v_4 \\ v_4 \end{pmatrix}} \to 0$$

Assume: Problem is correct

$$A = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}$$

$$B = \begin{pmatrix} v_1 \\ 4v_2 - 3v_1 \\ \vdots \end{pmatrix}$$

$$\det(A) = \det(A^T)$$

$$\begin{pmatrix} | & | & | & | \\ v_1 & v_2 + 7v_4 & v_3 - 2v_4 & v_4 \\ | & | & | & | \end{pmatrix}$$

$$= S \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | & | \\ Sv_1 & Sv_2 & Sv_3 & Sv_4 \\ | & | & | & | \end{pmatrix}$$

find S so that $S(v_1\ v_2\ v_3\ v_4) =$

$$\det S = 1$$

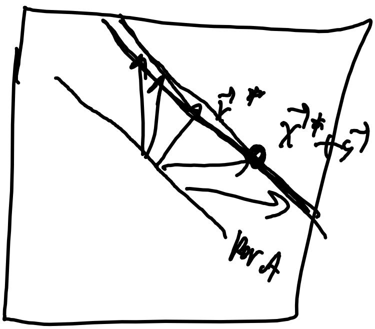
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 7 & -2 & 1 \end{pmatrix}$$

$$
\begin{tikzcd}
A \arrow[r, "R_2 \mapsto R_2 + 7R_4"] & B \arrow[d, "R_3 \mapsto R_3 - 2R_4"] \\
& C
\end{tikzcd}
$$

These row ops don't change det.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

least square solution $\underline{\vec{x}^*}$



$\vec{y}$ in ker A $\quad \vec{x}^* + \vec{y}$

$$A(\vec{x}^* + \vec{y}) = A\vec{x}^* + \cancel{A\vec{y}}^{\,0} = A\vec{x}^*$$

$$\ker A = \left\{ \begin{pmatrix} c \\ c \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$\vec{x}^*$ is $\underbrace{A\vec{x} = \vec{b}}$ least square

$$\Updownarrow$$

$$A^T A \vec{x}^* = A^T \vec{b}$$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$\left(\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}\right)^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

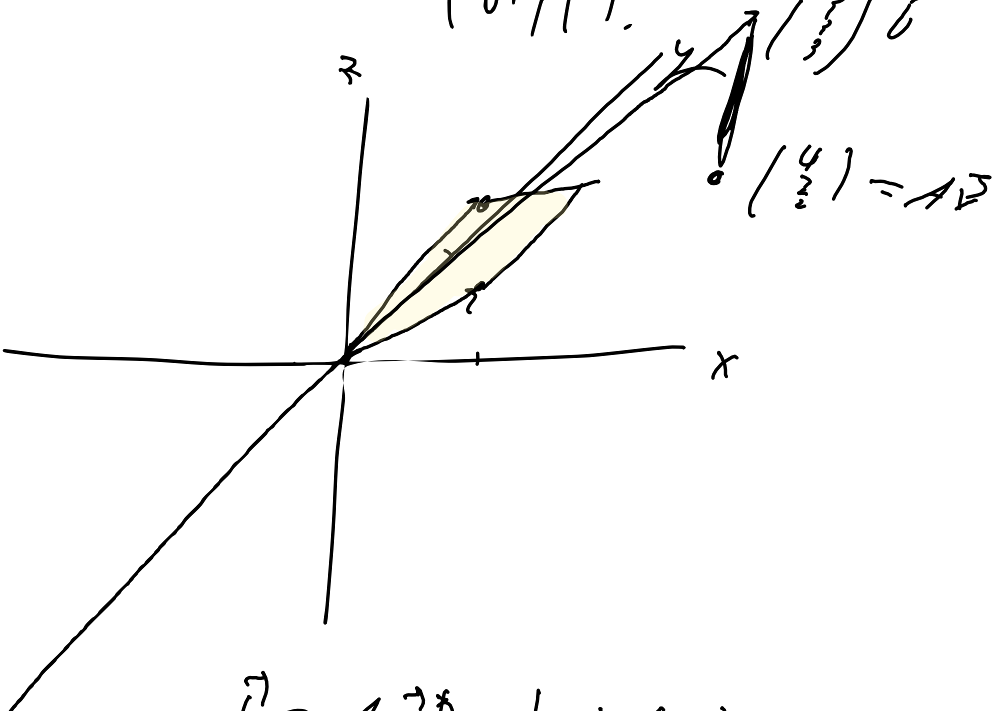
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4-2 \\ -2+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$A\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}1&1\\1&0\\0&1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}4\\2\\2\end{pmatrix}$$

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)$$



$$\underline{\vec{b} - A\vec{x}^* \perp \operatorname{im}(A)}$$

$\boxed{6.3.12}$

all 4x4 matrices w/ ±1, 0

max'l value of det

2x2

1,0

2^4 16

1 $(b, 0)$

4x4 16 entries

3^{16}

v_1 v_2 v_3 v_4

-1, 0, 1] matrices symmetric

n x n

LU

$A = QR$

$\det(A) = \det(Q)\det(R)$

$\det(Q^T) = \det(Q)$

$\frac{1}{\det(Q)}$

$\longleftrightarrow \det(Q) = \pm 1$

$$|\det(A)| = \sum_{\text{patterns } P} \underbrace{(\operatorname{sgn} P)}_{-1,1}\underbrace{(\operatorname{prod} P)}_{-1,0,1}$$

??

$$\le \cancel{24} \qquad \max X$$

$$A = QR$$

LU $\qquad A = \underbrace{SDS^{-1}}_{\text{diag}}$ $\qquad \therefore \det = \det D = \prod \text{diag}$

$$\|v+w\| \le \|v\| + \|w\|$$

equality: $v = \lambda w$

$$\|\operatorname{proj}_V x\| \le \|x\|$$

equality iff $x \in V$

$v_i = i^{\text{th}}$ col of A

$$\det R = |\det(A)|$$

$$\det R = \prod_{i=1}^{4} R_{ii} = \prod_{i=1}^{4} \|v_i^{\perp}\|$$

$R_{11} = \|v_1\|$

$R_{22} = \|v_2^{\perp}\|$

$R_{33} = \|v_3^{\perp}\|$

$R_{44} = \|v_4\|^{\perp}$

$$A = U \Sigma V$$

U: orthog $\qquad V$: orthog

$$\det R = \prod_{i=1}^{4} \|v_i^{\perp}\|$$

$v_i^{\perp}$??

$\neq$

v_i !!

$\pm 1, 0$

$$\|v_i^{\perp}\| \le \|v_i\|$$

when is this equality?

$v_2^{\parallel} = \mathrm{proj}_{v_1}(v_2)$

$$|\det(A)| = \det(R) = \prod_{i=1}^{4} \|v_i^{\perp}\| \le \prod_{i=1}^{4} \|v_i\| = \|v_1\|\|v_2\|\|v_3\|\|v_4\|$$

$$|\det(A)| \le \prod \|v_i\|$$

$$|\det A| \le \|v_1\|\,\|v_2\|\,\|v_3\|\,\|v_4\|$$

$\|v_i^\perp\| = \|v_i\|$?

v_i perp to $v_1, v_2, \ldots v_{i-1}$

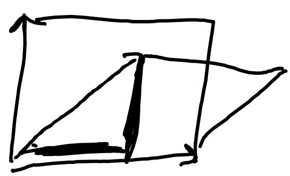
$\Downarrow$

$v_i^\perp = v_i$

$$\underline{|\det(A)|} \leq \underbrace{\|v_1\|}_{\leq 2}\underbrace{\|v_2\|}_{\leq 2}\underbrace{\|v_3\|}_{\leq 2}\underbrace{\|v_4\|}_{\leq 2} \leq \underline{16}$$

when equality?

$\boxed{\text{columns are perp}}$



$\cancel{\pm 16}$

look only @ those w/ perp columns

$\|v_i\| \leq 2$, all entries of v_i are non zero

$A \quad \pm 1, 0 \quad 4\times 4$

cols are perp

w/ zeros

$\Downarrow$

$|\det(A)| = 16$, maximum

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$