5.1 – 5.4, 6.1 – 6.3

5.1 Orthogonal projections / bases

Definitions

perpendicular $\vec{v} \cdot \vec{w} = 0$

length $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

unit length 1

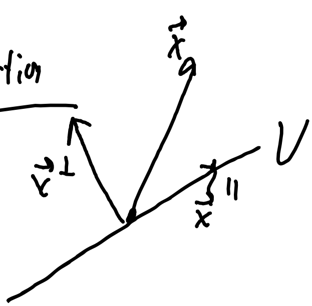
orthonormal $\vec{u}_1, \ldots, \vec{u}_m$ in $\mathbb{R}^n$

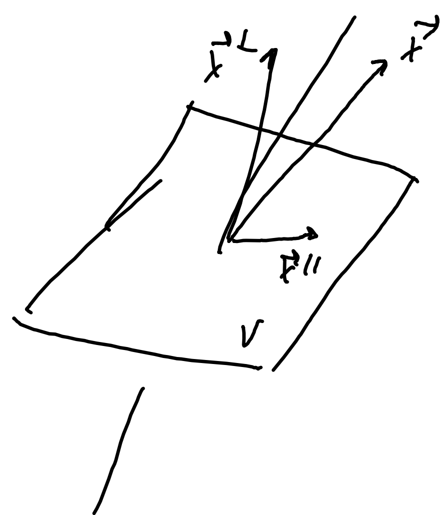
$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i = j \\ 0 & o/w \end{cases}$$

all unit / perp to another

Thought: Does orthonormal $\Rightarrow$ lin indep?

Projection





Thm. for $\vec{x}$ in $\mathbb{R}^n$

V a subspace of $\mathbb{R}^n$

$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ for $\vec{x}^{\parallel}$ in V

$\vec{x}^{\perp}$ perp to all vectors in V

uniquely

$T(\vec{x}) = \vec{x}^{\parallel}$ is called orthogonal projection onto V

$T = \mathrm{proj}_V$

This is linear

Let $\vec{u}_1, \ldots, \vec{u}_k$ be a basis for V

$\vec{x}^{\parallel} = \sum (\vec{x} \cdot \vec{u}_i) \vec{u}_i$

why does an ONB exist?

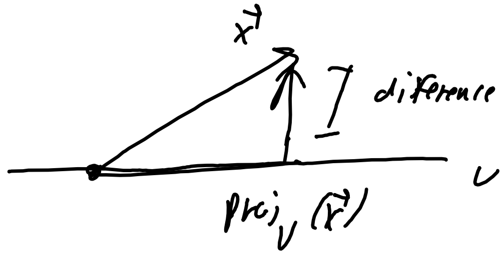
what is $\ker(\mathrm{proj}_V)$? $\mathrm{im}(\mathrm{proj}_V)$?

what is proj_V^2 ?

How do proj_V and $\mathrm{proj}_{V^\perp}$ relate? Their matrices?

Thm. $\|\text{proj}_V \vec{x}\| \leq \|\vec{x}\|$

equal if and only if $\vec{x}$ in V



Angles

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$$

$$\theta \text{ in } [-\pi, \pi]$$

5.2.

find ONB?

Gram-Schmidt / QR factorization

Thm. Let $\vec{v}_1, \ldots, \vec{v}_n$ be linearly inden
(<=> a basis for a subspace)

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{u}_2 = \frac{v_2^\perp}{\|v_2^\perp\|} \qquad \vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp} \text{ w.r.t. } \mathrm{Span}(\vec{v}_1)$$

$\vdots$

$$\vec{u}_i = \frac{v_i^\perp}{\|v_i^\perp\|} \qquad \vec{v}_i = \vec{v}_i^{\parallel} + \vec{v}_i^{\perp} \text{ w.r.t. } \mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_{i-1})$$

Then for all i, $\mathrm{Span}(\vec{u}_1, \ldots, \vec{u}_i) = \mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_i)$.

And the $\vec{u}_i$ are orthonormal.

Moral: can always extend a basis of a subspace V of $\mathbb{R}^n$ to all of $\mathbb{R}^n$.
with G-S, can convert bases to orthonormal ones.
Hence, can find and extend orthonormal bases of a subspace V of $\mathbb{R}^n$ to all of $\mathbb{R}^n$

QR factorization

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}}_{Q} R$$

R upper triangular

$R_{ij} = \vec{u}_i \cdot \vec{v}_j$ for $i < j$.

$$\text{so } R_{ii} = \vec{u}_i \cdot \vec{v}_i = \frac{\vec{v}_i^{\perp} \cdot (\vec{v}_i^{\parallel} + v_i^{\perp})}{\|\vec{v}_i^{\perp}\|} = \|\vec{v}_i^{\perp}\|$$

Moral: Gram-Schmidt + QR factorization are the same process, just written differently. All the same computations are done.

Orthogonal matrices / transformations

When is a matrix A orthogonal?
If any of the following hold;

i) $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x}$

ii) The columns of A are an orthonormal basis

iii) $A^{-1} = A^T$

iv) $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y}$

Idea: orthogonal matrices / transformations are "rigid", which is reflected in their length + angle preservation

(why do they preserve angles?)

Anything you can do w/ a piece of paper is orthogonal

Orthogonal projection revisited

$$Q = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{q}_k \\ | & & | \end{pmatrix}, \quad \vec{u}_1, \ldots, \vec{q}_k \text{ an ONB for } V.$$

Then $\mathrm{proj}_V = QQ^T$

why? check how it acts on the $\vec{q}_i$!

5.4 Least Squares

Transpose and orthogonal complements

$$(\operatorname{im} A)^{\perp} = \ker(A^T)$$

Recall that $(A^T)^T = A$, $(V^{\perp})^{\perp} = V$

Hence, $(\operatorname{im}(A^T))^{\perp} = \ker((A^T)^T)$

$= \ker(A)$

$$\operatorname{im}(A) = \ker(A^T)^{\perp}$$

$$\operatorname{im}(A^T) = \ker(A)^{\perp}$$

$$\ker(A) = \ker(A^T A)$$

We can't always solve systems, so we'll get as close as possible.

Thm. $\text{proj}_V(\vec{x})$ is the closest vector in V to $\vec{x}$.

Least squares: $\vec{x}^*$ is a least squares solution to $A\vec{x} = \vec{b}$

if $\|b - A\vec{x}^*\| \leq \|b - A\vec{x}\|$ for all $\vec{x}$

Hence, $A\vec{x}^* = \text{proj}_{\text{im}(A)}(\vec{b})$

$\Updownarrow$

$\vec{b} - A\vec{x}^*$ in $(\text{im} A)^\perp = \ker(A^T)$

$\Updownarrow$

$A^T\vec{b} = A^T A\vec{x}^*$

So $\vec{x}^*$ is a least squares sol to $A\vec{x} = \vec{b}$

$\Updownarrow$

$\vec{x}^*$ solves $A^T A\vec{x} = A^T\vec{b}$

Determinants (sorry, this part is rushed)

6.1

Def. $\det(A) = \sum_{\text{patterns } P} (\text{sgn } P)(\text{prod } P)$

If A upper Δ, $\det(A) = \prod a_{ii}$

6.2

$\det(A^T) = \det(A)$

$$T(\vec{x}) = \det \begin{pmatrix} - \vec{v_1} - \\ \vdots \\ - \vec{x} - \\ \vdots \\ - \vec{v_n} - \end{pmatrix} \text{ is linear}$$

Computing w/ row reduction

$A \xrightarrow{\text{swap}} B$ $\quad \det(B) = -\det(A)$

$A \xrightarrow{R_i \mapsto R_i + \lambda R_j} B$ $\quad \det(A) = \det(B)$

$A \xrightarrow{R_i \mapsto \lambda R_i} B$ $\quad \det(B) = \lambda \det(A)$

$\det(A) = \frac{1}{\lambda} \det(B)$

A invertible $\Leftrightarrow$ $\det A \neq 0$

$\det(AB) = \det(A)\det(B)$

A similar to $B \Rightarrow \det(A) = \det(B)$

$\therefore \det(T)$ makes sense irrespective of basis

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Expansion by minors

$A_{ij} =$ A upon deleting the i^{th} row & j^{th} column

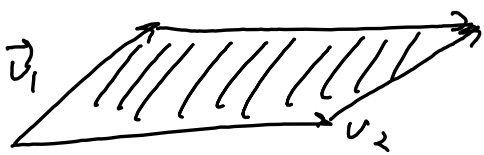
$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \quad \text{down col } j$$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \quad \text{along row } i$$

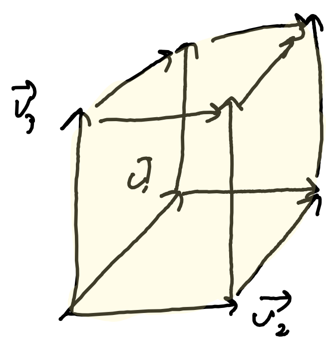
Geometry of det + Cramer's rule

Say $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$

det A = "signed" volume of the parallelepiped formed by $\vec{v}_1, \dots, \vec{v}_n$



area = det A



$$|\det(A)| = \frac{\text{volume of } T[S]}{\text{volume of } S}$$

Cramer's rule