We started w/ systems of linear equations.

$$x + y + z = 1$$
$$2x + 3y + 4z = 2$$
$$5x + 6y + 18z = 3$$

and simplified our notation to

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 2 \\ 5 & 6 & 18 & 3 \end{array}\right)$$

which makes computation vastly easier.

Now, let's recall the simplest possible linear system:

one equation and one variable.

$ax = b$.

How do we solve it? $x = \frac{b}{a}$ if $a \neq 0$

If $a = 0$, $b = 0$ any x works

$a = 0$ $b \neq 0$ inconsistent

We then turn our augmented matrices into a single equation via matrix algebra

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 18 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

which we write generally as $A\vec{x} = \vec{b}$. This is our equation w/ one variable $\vec{x}$. One way to solve is to "divide" by A, i.e. $\vec{x} = A^{-1}\vec{b}$. But this has subtleties, we'll see later.

Encompassed vast amounts of data into just one simple looking equation subtleties: $\vec{b}$ can't always be reached

$\vec{x}$ is not always unique

analyzing these leads us to images and kernels which are naturally understood in the context of linear transformations and the geometry of their subspaces.

we introduced linear transformations and their corresponding matrices.

$$T(\vec{x}) = A\vec{x}$$

i^{th} col. of A is $T(\vec{e_i})$.

Many geometric constructions are linear (take grids to grids) so they are well understood by using matrices.

scaling along an axis
projection
rotation
reflection
shearing

Some general geometric thoughts:

If I draw an evenly spaced grid and apply a linear transformation, what does the output look like? Draw some pictures for each of the above.

what is the image of a circle under a linear transformation?

We are led then to subspaces, especially the image and kernel of a linear transformation

Everything centers on the linear combination

$$T(a\vec{x}+b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$$

If $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v_n} \\ | & & | \end{pmatrix}$, then $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{v_1} + \dots + x_n \vec{v_n}$

columns

a subspace is then a subset, say V, of $\mathbb{R}^n$ for which we can take linear combinations of its elements and stay in V. That is, for $\vec{x}$, $\vec{y}$ in V and c a scalar,

$$\vec{x}+\vec{y} \text{ in } V$$
$$c\vec{x} \text{ in } V$$

The image and kernel are special subspaces attached to a linear transformation $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$

The image is the set of all outputs, so the set of all $\vec{b}$ in $\underline{\mathbb{R}^n}$ so that $T(\vec{x}) = \vec{b}$ is solvable

The kernel is the solutions $\vec{x}$ in $\underline{\mathbb{R}^m}$ to $T(\vec{x}) = \vec{0}$.
This measures the imprecision in solving $T(\vec{x}) = \vec{b}$.

(food for thought: If we have one solution $T(\vec{x_0}) = \vec{b}$, how do we describe all solutions in terms of $\vec{x_0}$ and $\ker(T)$?

To make this "measurement" idea formal, we introduce a notion of size — the dimension, which leads us to the incredible rank-nullity theorem. $m = \dim \operatorname{im} T + \dim \ker T$. For $\underline{\underline{\text{any}}}$ $T: \mathbb{R}^m \to \mathbb{R}^n$ linear!

Ch. 1

Start w/ systems of linear equations

$$2x + 3y + 4z = 1$$
$$x + y + z = 2$$

Incredibly, we can algorithmically solve these.

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -3 \end{array}\right) \xrightarrow{R_1 - R_2} \begin{array}{c} \left(\begin{array}{cccc} 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & -3 \end{array}\right) \\ \begin{array}{cccc} x & y & z & \end{array} \end{array}$$

$$x - z = 5$$
$$y + 2z = -3$$

$$x = z + 5$$
$$y = -2z - 3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

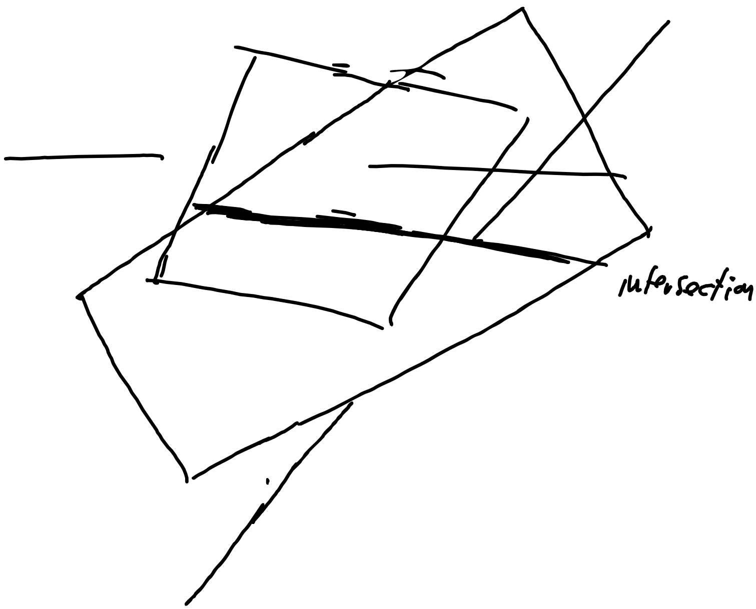
line.

Algebraic side vs geometric side

$2x + 3y + 4z = 1$ <-> plane in 3d space

$x + y + z = 2$ <-> plane in 3d space

solving = find intersection



This is essentially a solve problem via Gauss-Jordan elimination,

Keywords; Linear equations

coefs

vars

const. term

System of lin eqs. → geometric and algebraic meaning

consistent

inconsistent

Augmented matrix

row operations

ref, pivots/ lead 1s, rank

Ch. 2

Dictionary between linear transformations and matrices

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n \quad \longleftrightarrow \quad A$ an $n \times m$ matrix

$T(\vec{x}) = A\vec{x} \quad \longleftrightarrow \quad i^{th}$ col of $A = T(\vec{e}_i)$.

composition

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n \quad \longleftrightarrow \quad A \quad n \times m$

$S: \mathbb{R}^n \longrightarrow \mathbb{R}^k \quad \longleftrightarrow \quad B \quad k \times n$

$S \circ T: \mathbb{R}^m \longrightarrow \mathbb{R}^k \quad \longleftrightarrow \quad BA \quad k \times m$

(memorize formula or picture)

These encode lots of geometric transformations

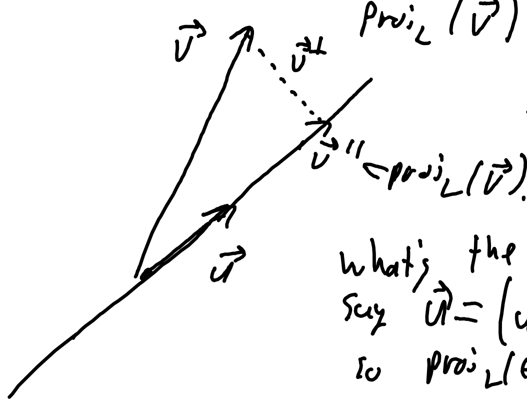
Scaling an axis $\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$

orthogonal projection onto L. $\quad \vec{u}$ a unit vector

$$\text{proj}_L(\vec{v}) = (\vec{v} \cdot \vec{u})\,\vec{u}$$

correlation between $\vec{v}$ and $\vec{u}$

"how much of $\vec{v}$" is in the $\vec{u}$ direction



what's the matrix? $\text{proj}_L(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u})\,\vec{u}$

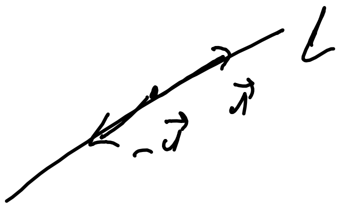
Say $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Then $\vec{e}_1 \cdot \vec{u} = u_1$

so $\text{proj}_L(\vec{e}_1) = \begin{pmatrix} u_1^2 \\ u_1 u_2 \end{pmatrix}$. similarly $\text{proj}_L(\vec{e}_2) = \begin{pmatrix} u_1 u_2 \\ u_2^2 \end{pmatrix}$

So matrix of proj_L is

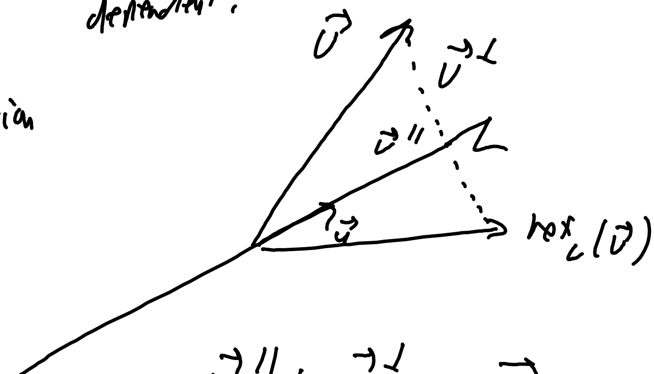
$$\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}$$

Note: what if we used $-\vec u$ instead? This is also a unit vector on L.

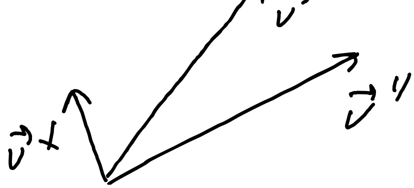


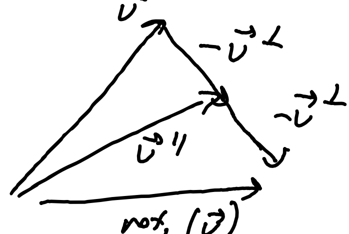
Can you see why these columns are linearly dependent?

reflection



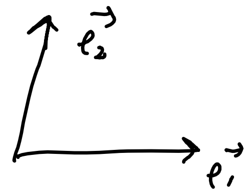
$$\vec v^{\parallel} + \vec v^{\perp} = \vec v$$

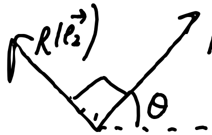




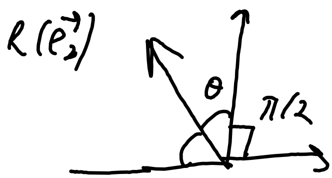
$$\mathrm{ref}_L(\vec v) = \vec v - 2\vec v^{\perp} = \vec v^{\parallel} - \vec v^{\perp} = \vec v^{\parallel} - (\vec v - \vec v^{\parallel})$$
$$= 2\vec v^{\parallel} - \vec v = 2\,\mathrm{proj}_L(\vec v) - \vec v$$

Rotation:





$R(\vec{e_1}) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$



so $R(\vec{e_2}) = \begin{pmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$

$R \longleftrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

observe: columns are length 1
and perpendicular
This makes sense, as R preserves length
and angle, and the columns are $R(\vec{e_i})$.

Invertibility

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ linear

A $m \times n$ matrix corresponding to T

The following are equivalent.

1) T is invertible as a function $\quad S = T^{-1}$
(there is $S: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ so that $T \circ S = S \circ T =$ identity)
$T(S(\vec{x})) = \vec{x}, \quad S(T(\vec{x})) = \vec{x}$

note: need only check one of those. This is a consequence of rank-nullity

2) A is an invertible matrix
(there is an $m \times m$ matrix B so that $AB = I_n$, $BA = I_n$)

$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$B = A^{-1}$

3) $\operatorname{rank}(T) = n$

4) $\operatorname{null}(T) = 0$

5) $\operatorname{im}(A) = \mathbb{R}^m$

6) $\ker(A) = \{\vec{0}\}$

7) $\operatorname{rref}(A) = I_n$

8) $T(\vec{x}) = \vec{b}$ has a solution for all $\vec{b}$

9) $T(\vec{x}) = \vec{b}$ has a unique solution for all $\vec{b}$

How to find inverse

$$\text{rref}\left(A \;\vdots\; I_n\right) = \left(I_n \;\vdots\; A^{-1}\right)$$

Essentially, solve
$$A\vec{x}_1 = \vec{e}_1$$
$$A\vec{x}_2 = \vec{e}_2$$
$$A\vec{x}_3 = \vec{e}_3$$

A^{-1} has columns $\vec{x}_1, \vec{x}_2, \vec{x}_3$

Ch. 3

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ linear

A its corresponding matrix

image: $\text{im}(T) =$ set of outputs

$=$ set of $\vec{b}$ so that $T(\vec{x}) = \vec{b}$ solvable

$= \{ T(\vec{x}) \mid \vec{x} \text{ in } \mathbb{R}^m \}$

"which equations can be solved?"

$\ker(T) =$ solutions to $T(\vec{x}) = \vec{0}$

$= \{ \vec{x} \text{ in } \mathbb{R}^m \mid T(\vec{x}) = \vec{0} \}$

"How unique are the solutions that exist?"

Both are <u>subspaces</u>, $\text{im}(T)$ a subspace of target, $\mathbb{R}^n$
range

$\ker(T)$ a subspace of input, $\mathbb{R}^m$
domain

A subset of $\mathbb{R}^n$ is a <u>subspace</u> if it contains the origin $\vec{0}$ and is "closed under linear combination"

 in V then  in V

How to find $\mathrm{im}(T)$, $\ker(T)$?

$\mathrm{im}(T)$ = span of columns of A

we can find a basis by omitting redundant vectors

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

either column is redundant.

$\ker(T)$? We're solving a system of linear equations!

$$\left(A \,\middle|\, 0 \right)$$

can rref this. also useful for image

redundant cols in $\mathrm{rref}(A)$ <-> redundant cols in A