- OH next monday changed
- final review (final Th 3pm)
- evals due Tu 8am

Inner products

$F = \mathbb{R}$ or $\mathbb{C}$ $\qquad \overline{a+bi} = a-bi$

V vector space $/F$,
an inner product on V is a function
$V \times V \longrightarrow F$ s.t. the following hold for
all $x, y, z \in V$, $c \in F$

i. $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
ii. $\langle cx, z \rangle = c\langle x, z \rangle$
iii. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
iv. $\langle x, x \rangle > 0$ for all $x \neq 0$ (pos. real #)

linear in slot 1
conj. linear in slot 2 $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$

e.g. – $V = F^n$

$$\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} a_i \overline{b_i}$$

$F = \mathbb{R}$, dot product

$$\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} a_i \overline{a_i}$$

$$= \sum |a_i|^2 > 0 \text{ if some } a_i \neq 0$$

– $V = F^\infty = \{$ sequences $\{a_n\}_{n=1}^{\infty}$ with $a_n \in F$ s.t. all but finitely many a_n are $0\}$.

$$\langle (a_n)_n, (b_n)_n \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n} \quad \text{(a finite sum)}$$

– $\ell^2(F) = \{$ seq. $(a_n)_{n=1}^{\infty}$ w/ $a_n \in F$ s.t. $\sum |a_i|^2$ convergent. $\}$

$$\langle (a_n)_n, (b_n)_n \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$$

(check convergence)

$C([0,1]) = \{ f : [0,1] \longrightarrow \mathbb{C} \quad \text{cont.} \}$

$$\langle f, g \rangle = \int_0^1 f(x)\, \overline{g(x)}\, dx \quad \in \mathbb{C}$$

$\boxed{L^2([0,1])} = \{ f : [0,1] \to \mathbb{C} \;/\; \int_0^1 |f(x)|^2 dx \ \text{converges} \}$

$$\langle f, g \rangle = \int_0^1 f(x)\, \overline{g(x)}\, dx \quad \text{converge}$$

$B^* = \overline{B^t}$

$\underline{M_{n\times m}(F)}$, $\quad \langle A, B \rangle = tr(B^* A)$

$m = 1$, $\quad M_{n\times 1}(F) = F^n$

$$\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) = tr\left(\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}^* \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) = tr\left((\overline{b_1} \cdots \overline{b_n}) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right)$$

$$= tr\left(\underline{\sum a_i \overline{b_i}} \right) \quad 1\times 1 \text{ matrix}$$

$$= \sum a_i \overline{b_i}$$

$$M_{1\times 1}(F) \xrightarrow[\text{iso}]{tr} F$$

invertible

$M_{n\times m}(F)$ $\quad \langle A, B\rangle = \mathrm{tr}(B^* A)$

wt. $\langle A, A\rangle > 0$ for $A \neq 0$

$$\langle A, A\rangle = \mathrm{tr}(A^* A) = \sum_i (A^* A)_{ii}$$

$$\|A\|^2 = \sum_{ij} |a_{ij}|^2$$

$$\begin{array}{cc} A^* & A \\ \begin{pmatrix} - \bar{v}_1 - \\ \vdots \\ - \bar{v}_m - \end{pmatrix} & \begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix} \end{array}$$

ij^{th} entry $\quad \bar{v}_i \cdot v_j = \langle v_j, v_i\rangle$

ii^{th} entry $\quad$ is $\quad \langle v_i, v_i\rangle = \|v_i\|^2$

$$= \sum_i \|v_i\|^2 \qquad v_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$$

$$= \sum_i \sum_j |a_{ij}|^2$$

$$A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}, \quad B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}$$

$$\|A\|^2 = 1^2 + |2+i|^2 + 3^2 + |i|^2$$
$$= 1 + 5 + 9 + 1$$
$$= 16$$

$$\|B\|^2 = |1+i|^2 + 0^2 + |i|^2 + |-i|^2$$
$$= 2 + 0 + 1 + 1$$
$$= 4$$

$$\langle A, B \rangle = tr(B^* A)$$
$$= tr\left(\begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}\right)$$
$$= tr\begin{pmatrix} 1-4i & ? \\ ? & -i \end{pmatrix} = 1-4i+(-1)$$
$$= -4i$$

$$\langle B, A \rangle = \overline{-4i} = 4i$$

on F^n, $\langle v, w\rangle = w^* v$

adjoint

$\langle Av, w\rangle = \langle v, A^* w\rangle$

$\langle Av, w\rangle = \sum_i (Av)_i \overline{w_i}$

$(Av)_i$

$=$

$\sum_j a_{ij} v_j$

$\sum_{ij} a_{ij} v_j \overline{w_j}$

$\langle v, A^* w\rangle$

$=$

$\sum_i v_i \overline{(A^* w)_i}$

$(A^* w)_i$

$= \sum_j A^*_{ij} w_j$

$= \sum_j \overline{A_{ji}} w_j$

$\sum_{ij} v_j A_{ji} \overline{w_j}$

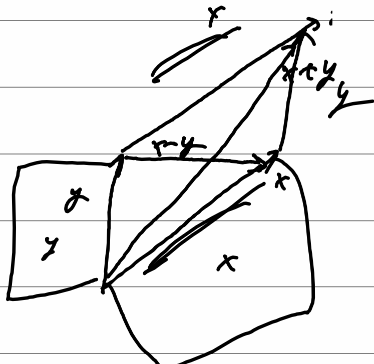
$\langle Av, w\rangle = w^* Av$

$= (A^* w)^* v$

$= \langle v, A^* w\rangle$

Parallelogram law

$x, y \in V$ an inner product space



$\sum$ squares of outside lengths

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$\sum$ squares of diagonals

Prop. $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$

Pf. RHS. $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$= \langle x,x\rangle + \cancel{\langle x,y\rangle} + \cancel{\langle y,x\rangle} + \langle y,y\rangle + \langle x,x\rangle + \cancel{\langle x,-y\rangle} + \cancel{\langle -y,x\rangle} + \langle -y,-y\rangle$

$\langle x,y\rangle + \langle x,-y\rangle = \langle x,y\rangle + (-\langle x,y\rangle) = 0$

$= \langle x,x\rangle + \langle y,y\rangle + \langle x,x\rangle + \overbrace{\langle -y,-y\rangle}^{(-1)\cdot(-1)\langle y,y\rangle} = \langle y,y\rangle$

$= 2\|x\|^2 + 2\|y\|^2$

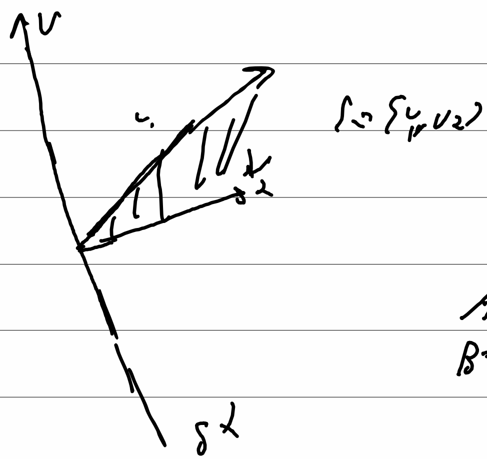
"polarization"

Orthogonal Complements

V inner product space

$S \subseteq V$ subset, we let $S^\perp = \{v \in V \mid \langle v, s \rangle = 0 \text{ all } s \in S\}$
"orthog complement of S" $\quad v \perp S$

Prop. $S^\perp = (\operatorname{span} S)^\perp$



$A \subseteq B$
$B^\perp \subseteq A^\perp$

Pf. "$\supseteq$" Let $\underbrace{v \in (\operatorname{span} S)^\perp}$, wts $v \in S^\perp$
$\underline{\text{for all } w \in \operatorname{span} S}$ $\langle v, w \rangle = 0$
$S \subseteq \operatorname{span} S$, so in particular, for $s \in S$, $\langle v, s \rangle = 0$
Thus, $v \in S^\perp$

"⊆" $S^\perp \subseteq \operatorname{span}(S)^\perp$

[Let $v \in S^\perp$, i.e. $\langle v, s\rangle = 0$ all $s \in S$

Let $w \in \operatorname{span}(S)$,

wts $\langle v, w\rangle = 0$.

$w = \sum a_i s_i$ $a_i \in F$, $s_i \in S$,

?

wts $\langle v, w\rangle = 0$

$\langle v, w\rangle = \langle v, \sum a_i s_i\rangle$

$= \sum \overline{a_i} \langle v, s_i\rangle \to 0$ $s_i \in S$

$= 0$

$\therefore S^\perp = (\operatorname{span}(S))^\perp$

Thm. (Gram - Schmidt)

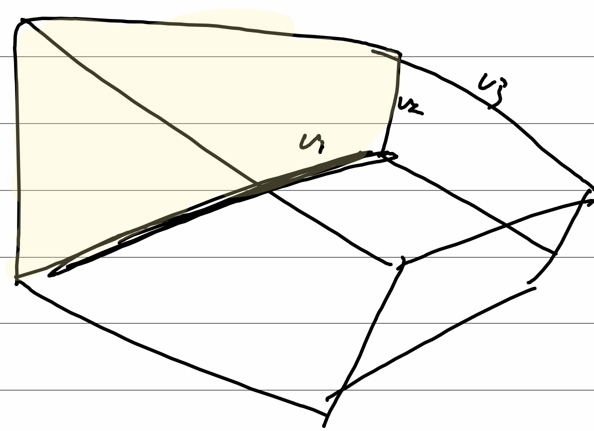
V inner product space

$\{v_1, \ldots, v_n\}$ lin indep in V $\quad$ ($e_1, \ldots, e_n$ in F^n)

there is $\{w_1, \ldots, w_n\}$ orthonormal $\left(\langle w_i, w_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}\right)$

s.t.

$$\operatorname{span}(v_1, \ldots, v_i) = \operatorname{span}(w_1, \ldots, w_i)$$



Application. V fd inner product space,

$W \leq V$ subspace, $W \oplus W^\perp = V$

pf. $\{w_1, \ldots, w_k\}$ basis for W

extend to $\{\underbrace{w_1, \ldots, w_k}_{W}, \underbrace{w_{k+1}, \ldots, w_n}_{U}\}$ basis for V

$W \oplus U = V$

a.s

$\{u_1, \ldots, u_k\}^{\perp} = W^{\perp}$

$\{\underbrace{u_1, \ldots, u_k}_{\substack{\text{spans } W \\ \text{ON basis for } W}}, \underbrace{u_{k+1}, \ldots, u_n}_{\substack{\text{ONB} \\ \text{for } W^{\perp}}}\}$ ONB for V

$$W \oplus W^{\perp} = V$$

$\langle u_{k+1}, u_1 \rangle = 0$

$\langle u_{k+1}, u_2 \rangle = 0$

$\vdots$

$\langle u_{k+1}, u_k \rangle = 0$