5, 2, 7

$A = PDP^{-1}$

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_n^n \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+y \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_{n-2} \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}$

A

$A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ <-> formula for n^{th} fib #

$A^3 = (PDP^{-1})^3 = PD\underbrace{P^{-1}P}_{I}D\underbrace{P^{-1}P}_{I}DP^{-1}$

$= PDDDP^{-1}$

$= PD^3P^{-1}$

$A^n = (PDP^{-1})^n = \underbrace{PD\overset{I}{(P^{-1}P)}D\overset{I}{(P^{-1}P)}\ldots\overset{I}{(P^{-1}P)}D}_{n}P^{-1}$

$= PD^nP^{-1}$

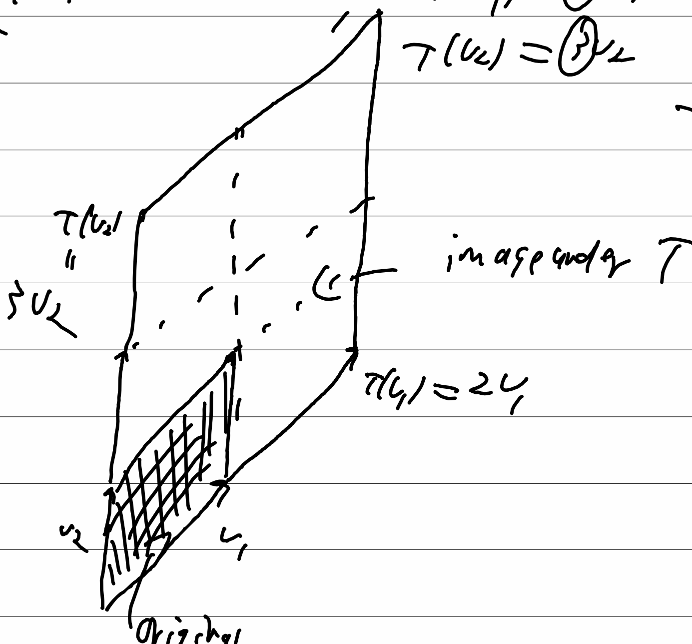
Relationship between eigenvalues + determinant

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(v_1) = 2 v_1$ $v_1, v_2 \neq 0$

$T(v_2) = 3 v_2$

$\to \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$



area of big = 6 · area of small

$\det(T) = 6$

$2 \cdot 3 = 6$

eigenvalue det.

$$T: V \longrightarrow V$$

Conjecture, $\det(T) = \prod_{i=1}^{n} \lambda_i$

T diagonalizable, this holds

$$\det \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \prod \lambda_i$$

eigenvalues

if T is similar to an upper Δ matrix

$$_{\beta}[T]_{\beta} = \begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

$\det(T) = \prod a_{ii}$ &

$\underbrace{\text{eigenvalues of } T \text{ are the } a_{ii}}$

char poly $= \prod_{i=1}^{n} \underbrace{(a_{ii} - \lambda)}$.

Prop. $A \in M_{n\times n}(\mathbb{C})$, A similar to an upper triangular matrix

Rmk. We only need char poly of A to split

Pf. Key Fact. all nonconstant polys / $\mathbb{C}$ have a root

$\Downarrow$

char poly of A has a root

$\Downarrow$

A has an eigenvalue λ.

associated eigenvector $v \neq 0$

$Av = \lambda v$.

$\beta = \{v, \underbrace{v_2, \ldots, v_n}_{W}\}$ basis $[\lambda v]_\beta = \begin{pmatrix}\lambda \\ 0 \\ \vdots \\ 0\end{pmatrix}$

$$[A]_\beta^\beta = \begin{pmatrix} \lambda & \overbrace{* \; \cdots \; *}^{n-1} \\ 0 & \\ \vdots & \underbrace{B}_{n-1} \\ 0 & \end{pmatrix}$$

want upper Δ.

By induction $\begin{pmatrix}\lambda & * \\ 0 & *\end{pmatrix}$

assume that result

holds for $n-1 \times n-1$ matrices

$$[A]_\beta^\beta = \begin{pmatrix} \lambda & * & * & * \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

$B \in M_{n-1 \times n-1}(F)$, by inductive hypothesis, there is some $P \in M_{n-1 \times n-1}(F)$ invertible so that PBP^{-1} is upper triangular

$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & P & \\ 0 & & & \end{pmatrix}$$

$n \times n$ / F invertible

$$Q^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & P^{-1} & \\ 0 & & & \end{pmatrix}$$

$G\,[A]_\beta^\beta\, G^{-1} =$

$$\underbrace{\begin{pmatrix} 1 & c \;\cdots\; 0 \\ 0 & \\ \vdots & \rho \\ 0 & \end{pmatrix}}_{G} \underbrace{\begin{pmatrix} \lambda & ? \\ 0 & \\ \vdots & \beta \\ 0 & \end{pmatrix}}_{[A]_\beta^\beta} \underbrace{\begin{pmatrix} 1 & c \;\cdots\; 0 \\ c & \\ \vdots & \rho^{-1} \\ 0 & \end{pmatrix}}_{G^{-1}}$$

$$= \begin{pmatrix} \lambda & ? \\ 0 & \\ 0 & \rho\beta \\ 0 & \end{pmatrix} \begin{pmatrix} 1 & c \;\cdots\; 0 \\ c & \\ \vdots & \rho^{-1} \\ 0 & \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 \\ c \\ \vdots \\ 0 \end{pmatrix} = 0$$

$$= \begin{pmatrix} \lambda & ? \;\; ? \;\; ? \\ 0 & \\ \vdots & \rho\beta\rho^{-1} \\ 0 & \end{pmatrix}$$

$\rho\beta\rho^{-1} \to$ upper Δ

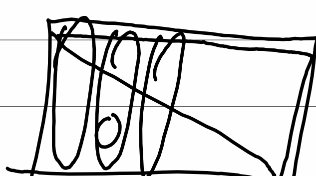
$$A \sim \begin{pmatrix} \lambda & * \\ 0 & \underbrace{PBP^{-1}}_{\text{upper } \Delta} \\ \vdots & \\ 0 & \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix} \quad \text{upper } \Delta$$

Recall. $T: V \to V$ similar to upper Δ

$\iff$

[there is a basis $\{v_1, \dots, v_n\}$ s.t. $T(v_i) \in \operatorname{span}(v_1, \dots, v_i)$]



$v_1 \neq 0$

$/\mathbb{C}$ $\therefore$ $T(v_1) = \lambda_1 v_1$ eigenvector

find v_2 lin. indep from v_1 s.t. $T(v_2) \in \mathrm{span}(v_1, v_2)$

$$T(v_2) = a_1 v_1 + a_2 v_2$$

$$T(v_2) - a_1 v_1 = a_2 v_2$$

$P : V \longrightarrow V$ projection onto $\mathrm{span}(v_1)$

$$V = \mathrm{span}(v_1) \oplus W$$

$$(T - P)(v_2) = T(v_2) - P(v_2) \leftarrow \mathrm{span}(v_1)$$

$$= T(v_2) - b_1 v_1 \quad \text{some } b_1$$

wts $(T-P)$ has an eigenvector, true as $/\mathbb{C}$

$$(T-P)v_2 = \lambda_2 v_2$$

$$\underbrace{T(v_2)} - \underbrace{b_1 v_1 = \lambda_2 v_2}$$

$$T(v_2) \in \mathrm{span}(v_1, v_2)$$

$\{v_1, \ldots, v_i\}$ already

$W = \operatorname{span}\{v_1, \ldots, v_i\}$

p projection onto W

eigenvector $(T - p)(v_{i+1}) = \lambda_{i+1} v_{i+1}$

$$\underbrace{T(v_{i+1}) = \underbrace{p(v_{i+1})}_{v_1, \ldots, v_i} + \lambda_{i+1} \underbrace{v_{i+1}}_{v_{i+1}}}_{\in \operatorname{span}(v_1, \ldots, v_{i+1})}$$

Fact. $T : V \longrightarrow V$, V fdvs / F

char poly of T split

$\Downarrow$

T similar to an upper $\triangle$ matrix roots of char poly

Applications.

i. $\det(T) = \prod_{i=1}^{n} \lambda_i$ $\lambda_1, \dots, \lambda_n$ eigenvalues w/ alg mult

"det is prod of eigenvalues"

$\mathcal{B}$ s.t. $[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\det(\;) = \prod \lambda_i$

$\det(\lambda I - T)\big|_{\lambda = 0} \to \det(-T) = (-1)^n \det(T)$

$\underbrace{\lambda I - T = \prod(\lambda - \lambda_i)}$

$\lambda = 0 \longrightarrow \prod(-\lambda_i) = (-1)^n \prod \lambda_i$ //

ii. $\operatorname{tr}(A) = \sum a_{ii}$

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$\Downarrow$

$\operatorname{tr}(PAP^{-1}) = \operatorname{tr}(A)$

$\begin{pmatrix} a_{11} & - & - & | \\ | & \ddots & & | \\ | & & \ddots & | \\ | & - & - & a_{nn} \end{pmatrix}$

"trace is Σ of eigenvalues"

$\operatorname{tr}([T]_{\mathcal{B}}^{\mathcal{B}}) = \sum \lambda_i$

$\quad\shortparallel$

$\operatorname{tr}(T)$ //

$(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$

$$\det(\lambda I - T) = \prod_i (\lambda - \lambda_i)$$

coefficient of $\underline{\lambda^{n-1}}$

$\underline{-\sum \lambda_i}$

$= -\mathrm{tr}(T)$

$$\begin{pmatrix} \lambda^n - (\mathrm{tr} A)\lambda^{n-1} \\ \vdots \\ +(-1)^n \det(A) \end{pmatrix}$$

λ^n coeff is 1

A 2×2, char poly of $A = \lambda^2 - (\mathrm{tr} A)\lambda + \det(A)$

$\sim$ $T: V \longrightarrow V$ whose char poly splits

$$[T]_\beta^\beta = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ e.v. (T)

$$[T^2]_\beta^\beta = \begin{pmatrix} \lambda_1^2 & & * \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix}$$ e.v. (T^2)

ev of T^2 are squares of eigenvalues of T

$$\begin{bmatrix} \lambda_1 & \cdots & \lambda_n & T \\ p(\lambda_1) & \cdots & p(\lambda_n) & T \end{bmatrix}$$

p polynomial $\quad p = \sum a_i x^i$

$$p(T) = \sum a_i \underbrace{T^i}_{\text{composition}}$$

$$[p(T)]_{\beta}^{\beta} = p\left([T]_{\beta}^{\beta}\right) = \begin{pmatrix} p(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & p(\lambda_n) \end{pmatrix}$$

e.v. of $p(T)$

$$Tv = \lambda v$$

$$(p(T))v = \sum a_i T^i v$$

$$= \sum a_i \lambda^i v$$

$$= p(\lambda) v$$

eigenvalues of T^{-1} are inverses of eigenvalues of T

$$[T^{-1}]_{\beta}^{\beta} \quad \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix} \to \text{e.v. of } T^{-1}$$

$\exp(T)$ has e.v. $\exp(\lambda)$, λ also e.v. of T

Application.

$$\begin{cases} A \text{ in } M_{3\times3}(\mathbb{R}) \subseteq M_{3\times3}(\mathbb{C}) \\ AA^t = I \\ \det(A) = 1 \end{cases}$$

$\Downarrow$

A rotation about an axis

Pf. $AA^t = I$, $A^t = A^{-1}$

$\{\text{ev. of } A^t\}$ $\{\text{ev. of } A^{-1}\}$

$\{\text{ev. of } A\}$ $1/\{\text{ev. of } A\}$

$\{\lambda_1, \lambda_2, \lambda_3\} = \{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\}$

$|\lambda_i| = 1$

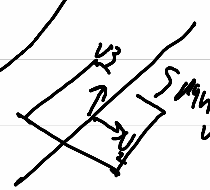
3×3, char poly is deg 3

$\therefore$ λ_1 is real.

$\lambda_1\lambda_2\lambda_3 = 1$ $\quad \lambda_3 = \overline{\lambda_2}$

$\lambda_1 \lambda_2 \overline{\lambda_2} = 1$, $\lambda_1 \cdot |\lambda_2|^2 = 1$

$\lambda_1 = 1$



$Av = v$

$\lambda_2 = e^{i\theta} \quad \lambda_3 = e^{-i\theta}$