OH Th 3-4 PM

$P_2(\mathbb{R})$

$\beta = \{x^2 - x, x^2 + 1, x - 1\}$

$\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$

$$\beta' \xrightarrow{\text{Change of coords}} \beta$$

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$[p]_{\beta'} \longmapsto [p]_{\beta}$$

$[I]_{\beta'}^{\beta}$ ⟸ Change of coord matrix $\beta' \to \beta$

$[T]_{\beta}^{\gamma}$

$[I]_{\beta'}^{\beta} [p]_{\beta'} = [p]_{\beta}$

$\beta = \{x^2 - x, x^2+1, x-1\}$

3×3 $\quad [I]_{\beta'}^{\beta}$

$\beta' = \{x^2+x+4, 4x^2-3x+2, 2x^2+3\}$

first column,

$[I]_{\beta'}^{\beta} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = [x^2+x+4]_{\beta'}$

first col of $[I]_{\beta'}^{\beta}$

is $[I]_{\beta'}^{\beta} [x^2+x+4]_{\beta'}$

$\parallel$

$[x^2+x+4]_{\beta}$

$x^2+x+4 = a(x^2-x)+b(x^2+1)+c(x-1)$

$1 \quad -2$

$[x^2+x+4]_{\beta} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, 1^{st} col of $[I]_{\beta'}^{\beta}$

$\boxed{x^2+x+4 = (a+b)x^2 + (-a+c)x + (b-c)\cdot 1}$

$a+b=1$, $\quad -a+c=1$, $\quad b-c=4$

$x^2 \qquad x \qquad 1$

$$\begin{array}{ccc|c} a & b & c & \\ \hline \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & & & -2 \\ & 1 & & 1 \\ & & 1 & 1 \end{array}\right)$$

first column of $[I]_{\beta'}^{\beta}$

$$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ solves}$$

$[x^2+x+4]_\beta$

2nd col. $[4x^2-3x+2]_{\beta'} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ -1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right)$

2nd column $= \underline{[I]}_{\beta'}^{\beta} [4x^2-3x+2]_{\beta'} = [4x^2-3x+2]_\beta$

this matrix

3rd col $\quad [2x^2+3]_\beta$

$\beta' = \{w_1, \dots, w_4\}$

$$[I]_{\beta'}^{\beta} = \begin{pmatrix} | & | & & | \\ [w_1]_\beta & [w_2]_\beta & \cdots & [w_n]_\beta \\ | & | & & | \end{pmatrix}$$

$\begin{pmatrix} 13 \\ 24 \end{pmatrix}$ $S = \{x^2, x, 1\}$ $\beta' = \{x^2+x+4,\ 4x^2-3x+2,\ 2x^2+3\}$

$$[I]_{\beta'}^{\beta} = [I]_S^{\beta} \cdot [I]_{\beta'}^{S}$$ $\beta = \{x^2-x,\ x^2+1,\ x-1\}$

$$\begin{tikzcd} \beta' \arrow[r] \arrow[dr] & \beta \\ & S \arrow[u] \end{tikzcd}$$

$$\begin{pmatrix} 1 & 4 & 2 \\ 1 & -3 & 0 \\ 4 & 2 & 3 \end{pmatrix}$$

$$\left([I]_{\beta}^{S}\right)^{-1}$$

$$[I]_{\beta}^{S} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 & 2 \\ 1 & -3 & 0 \\ 4 & 2 & 3 \end{pmatrix} \longleftrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 4 & 2 \\ -1 & 0 & 1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 4 & 2 & 3 \end{array}\right)$$

Determinants

choice of basis

$\det: L(V,V) \longrightarrow F$

$\det: M_{n\times n}(F) \longrightarrow F$

$T: V \longrightarrow V$, V fdvs

$\underbrace{\det(T)} = \det\left([T]_\beta^\beta\right)$ for some basis β

Q? why is this independent of that choice?

Prop. β, γ bases of V

$$\det\left([T]_\beta^\beta\right) = \det\left([T]_\gamma^\gamma\right)$$

Pf.

$$\underbrace{[I]_\beta^\gamma [T]_\beta^\beta [I]_\gamma^\beta} = [T]_\gamma^\gamma$$

$\longrightarrow [x]_\gamma$ $\qquad [T]_\gamma^\gamma [x]_\gamma$

$= [I]_\beta^\gamma [T]_\beta^\beta [x]_\beta$ $\qquad = [T(x)]_\gamma$

$= [I]_\beta^\gamma [T(x)]_\beta$

$= [T(x)]_\gamma$

$\beta \to \gamma$

$[I]_\beta^\gamma = \left([I]_\gamma^\beta\right)^{-1}$ $\qquad (\gamma \longrightarrow \beta)^{-1}$

$$\left([I]_\gamma^\beta\right)^{-1} \underline{[T]_\beta^\beta}\, [I]_\gamma^\beta = \underline{[T]_\gamma^\gamma}$$

$$\det(\;\downarrow\;) = \underline{\det}([T]_\gamma^\gamma)$$

$$\parallel$$

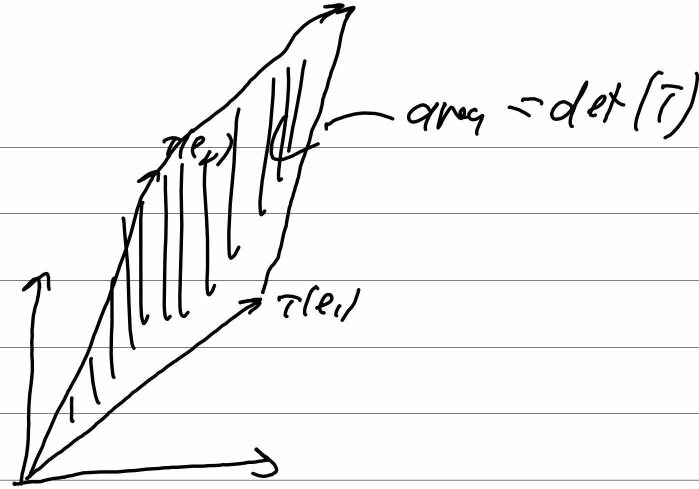
$$\underline{\det\left(\left([I]_\gamma^\beta\right)^{-1}\right)} \det\left(\underline{[T]_\beta^\beta}\right) \det\left([I]_\gamma^\beta\right) = \det\underline{\left([T]_\gamma^\gamma\right)}$$

$$= \frac{1}{\underline{\det\left([I]_\gamma^\beta\right)}} \det\left([T]_\beta^\beta\right) \underline{\det\left([I]_\gamma^\beta\right)}$$

$$=$$

$$= \det\left(\underline{[T]_\beta^\beta}\right)$$

$$\det(PAP^{-1}) = \det(A)$$



$$\left[\begin{array}{l} S \subseteq \mathbb{R}^2 \\ \text{area}(T(S)) = \underline{\det(T)} \cdot \text{area}(S) \end{array}\right]$$

$\det(T) \neq 0 \iff T$ invertible

Properties

- A upper triangular $n \times n$

$$A = \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

fact. $\det(A) = a_{11} a_{22} \cdots a_{nn}$

"pf" $\det(A) = a_{11} \det \underbrace{\begin{pmatrix} a_{22} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}}_{\text{induction}}$

- compute using rref

$i \neq j$

$R_i \mapsto R_i + \lambda R_j$

$A \longrightarrow B$

$$\underbrace{\begin{pmatrix} \vdots \\ - R_i - \\ \vdots \end{pmatrix}}_{A} \mapsto \underbrace{\begin{pmatrix} \vdots \\ - R_i + \lambda R_j - \\ \vdots \end{pmatrix}}_{B}$$

$\det(A) = \det(B)$

$$\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array}\begin{pmatrix} \vdots \\ 1 \\ \vdots \\ \lambda \\ \vdots \end{pmatrix}$$

$$\underbrace{\begin{array}{cc} & j \\ i & \begin{pmatrix} 1 & & & \\ & 1 & \lambda & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{array}}_{E}\ \underbrace{\begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_n & - \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i + \lambda R_j & - \\ & \vdots & \\ & R_n & \end{pmatrix}}_{B}$$

det = 1 as upper Δ

$\underbrace{E}_{\text{apply row op to } I} A = B$, $\det(B) = \det E \ \det A$
$= \det A$

$R_i \mapsto R_i + \lambda R_j$, $i \neq j$

$A \longrightarrow B$

Rmk. $\underline{\det(A) = \det(B)}$

$$A \xrightarrow{R_i \leftrightarrow R_j} B$$

$$E = \begin{array}{c|c} & \begin{matrix} & & & j & & & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ j \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & & 1 & & \\ & & & \ddots & & & \\ & 1 & & & & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

I swapping rows i and j

$\det(E) = -1$

$EA = B$, $-\det(A) = \det(B)$

Rule 2. $A \xrightarrow{\text{swap rows}} B$

$\underline{-\det(A) = \det(B)}$

$$A \xrightarrow{R_i \mapsto \lambda R_i} B$$

$$E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

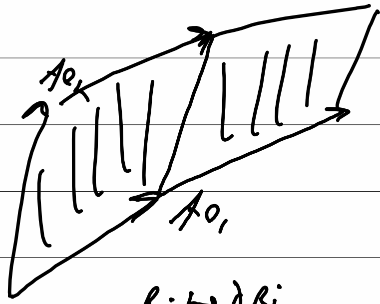
$\det = \lambda$

$EA =$ A w/ i^{th} row multiplied by λ

$\det(EA) = \det(B)$

$\|$

$\lambda \det(A)$ //



Rule 2. $A \xrightarrow{R_i \mapsto \lambda R_i} B$

$$\det(B) = \lambda \det(A)$$

$V = P_3(\mathbb{R})$

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (a_0 - a_2)\cdot 1 + (-a_0 + a_1 + a_3)x + (a_0 + a_1 - a_3)x^2 - a_2 x^3$$

$T: V \longrightarrow V$

S standard basis $\{1, x, x^2, x^3\}$

$$[T]_S^S = \begin{array}{c} \begin{array}{cccc} [T(1)]_S & [T(x)]_S & [T(x^2)]_S & [T(x^3)]_S \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{array}$$

$R_2 + R_1$, $R_3 - R_1$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$R_3 - R_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 0 \end{pmatrix} \longrightarrow R_4 + \frac{1}{2} R_3$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det\left([T]_{S}^{S'}\right) = \det(\downarrow)$$

$$= 1 \cdot 1 \cdot 2 \cdot 1 = 2$$

$$\det(T) = 2.$$

Diagonalizability

$T: V \to V$ fdvs diagonalizable, i.e. we have a basis β s.t. $[T]_\beta^\beta$ diagonal "$\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$"

2 ways this can fail

go to $\mathbb{C}$

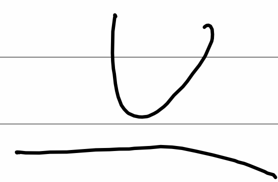
i. Char poly doesn't split

i.e. not enough roots to get eigenvalues

$A = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ char poly = λ^2+1

no real roots

not split over $\mathbb{R}$



$T: \mathbb{R}^2 \to \mathbb{R}^2$ 90° ccwise rotation

over $\mathbb{C}$ this has 2 roots $i, -i$

T 2 eigenvalues $i, -i$

diagonalizable not over $\mathbb{R}$, but over $\mathbb{C}$

ii. $gm(\lambda) < am(\lambda)$

$$A = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \quad n \times n$$

$$gm(\lambda) = \dim \ker (A - \lambda I)$$

$$= \dim \ker \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

$$= 1.$$

$$\wedge$$

$$am(\lambda) = n.$$

char poly of A $= \det \begin{pmatrix} \lambda - x & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - x \end{pmatrix}$

$$= (\lambda - x)^n$$

A not diagonalizable

$(n > 1)$