$V = P(\mathbb{R})$

$$D : V \longrightarrow V$$
$$p \longmapsto p'$$

$D(2x^2+1) = 4x$

$L(V, V)$

Claim: $\{I, D, D^2, \dots, D^n\}$ lin indep for all n

$(\Leftrightarrow \{I, D, D^2, \dots\}$ lin indep$)$

Pf. Let $\displaystyle\sum_{i=0}^{n} a_i D^i = 0$ ← 0 transformation $v \mapsto 0$

$(2^0 = 1)$

Convention. $D^0 = I$

$$a_0 I + \sum_{i=1}^{n} a_i D^i = 0$$

constant in p

$$\sum_{i=0}^{n} a_i D^i = 0 \in L(V, V)$$

const in p

$$\left(\sum_{i=0}^{n} a_i D^i\right)(p) = 0 \in P(\mathbb{R})$$

def of $L(V, V)$

$$= \sum a_i \underbrace{D^i(p)}_{p^{(i)}}$$

Fix p,

$$\begin{array}{ccc} L(V,V) & & T \\ \downarrow \mathrm{ev}_p & & \downarrow \\ V & & T(p) \end{array}$$

[for all $p \in P(\mathbb{R})$, $p^{(0)} = p$

$$\sum_{i=0}^{n} a_i \underbrace{p^{(i)}}_{} = 0$$]

goal. show $\underline{a_i = 0}$ for all i

Suggestion. plug in p

$p = 1$

$$\sum_{i=0}^{n} a_i \, p^{(i)} = 0$$

$$\underbrace{p^{(i)}}_{} = \begin{cases} 1 & i=0 \\ 0 & i=1 \\ 0 & i=2 \\ \vdots & \vdots \\ 0 & i=n \end{cases} \in P(\mathbb{R})$$

$P(\mathbb{R})$

$a_0 \cdot 1 + a_1 \cdot 0 + \cdots + a_n \cdot 0 = 0$

$a_0 \cdot \underbrace{1}_{P(\mathbb{R})} = 0$

$\underline{a_0} = 0$

$$\sum_{i=1}^{n} a_i p^{(i)} = 0$$

$p=1 \Rightarrow \underline{a_0 = 0}$

$p=x \Rightarrow \underline{a_1 = 0}$

$p=x^2 \Rightarrow a_2 = 0$

$p = x^{r}$

$\vdots$

$x^i \Rightarrow a_i = 0$

$\Downarrow$

all $a_i = 0$

$p^{(1)} = 1$

$p^{(2)} = 0$

$p^{(3)} = 0$

$\vdots$

$p^{(n)} = 0$

$$a_1 \cdot 1 + a_2 \cdot 0 + \cdots + a_n \cdot 0 = 0$$

$$a_1 \cdot 1 = 0 \quad \Rightarrow a_1 = 0$$

$$\sum_{i=2}^{n} a_i p^{(i)} = 0$$

$p = x^2$ $\quad p^{(2)} = 2, \; p^{(3)} = 0, \ldots, p^{(n)} = 0$

$$2 \cdot a_2 + a_3 \cdot 0 + \cdots + a_n \cdot 0 = 0$$

$\Downarrow$

$a_2 = 0$

$$\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & a_0' \\ & 0 \end{pmatrix}$$

$$D \longleftrightarrow \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 2 & & & 0 \\ & & 0 & 3 & & \\ & & & \ddots & \ddots & \\ & 0 & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

$$D^2 \longleftrightarrow \begin{pmatrix} 0 & 0 & 2 & & \\ & 0 & 0 & 6 & \\ & & \ddots & \ddots & \ddots \\ & & & 0 & \ddots \\ & & & & 0 \ \ddots \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 0 & 0 & 0 & 6 & 24 & \ddots \\ & \ddots & \ddots & 0 & 0 & \ddots \\ & & & \ddots & \ddots & \\ & & 0 & \ddots & & \\ & & & \ddots & & \end{pmatrix}$$

Def. Let $V = W_1 \oplus W_2$]

$T: V \longrightarrow V$ given by

$v = w_1 + w_2$, then $T(v) = w_1$.

This is called projection onto W_1 along W_2

fact. Every $v \in V$

is uniquely written as $w_1 + w_2$

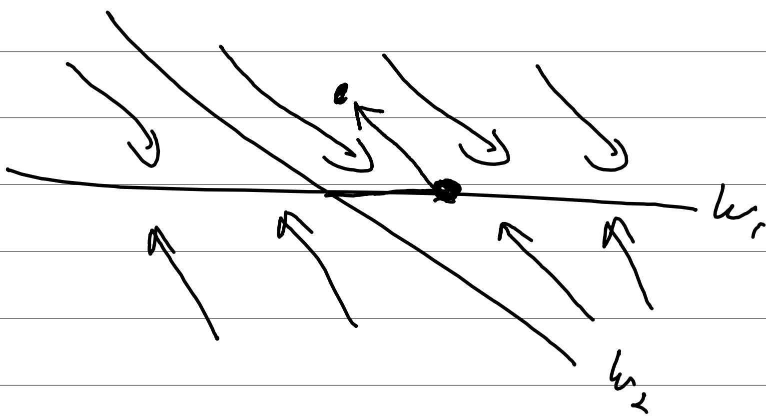
$w_1 \in W_1$

$w_2 \in W_2$

$$V = W_1 \oplus W_1$$

$$\Updownarrow$$

$$v = w_1 + w_2$$



$T \circ T$

$\uparrow$

Prop. $T^2 = T \iff T$ is the projection onto $W_1 = \{y \in V \mid T(y) = y\}$, along $\ker(T) = \{y \in V \mid T(y) = 0\}$

Pf. ($\Leftarrow$) Let T be projection onto W_1 along $\ker(T)$

wts $T^2 = T$.

Let $v \in V$. $V = W_1 \oplus \ker(T)$

$v = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in \ker(T)$.

$T(v) \overset{def}{=} w_1$.

$T^2(v) = T(T(v)) = \underline{T(w_1)} = \underset{W_1}{?} + \underset{\ker(T)}{?}$

$T^2(v) = T(w_1) = w_1 = T(v)$

$\underline{w_1} \in W_1 = \{y \in V \mid \underline{T(y) = y}\}$

$T(w_1) = ?$
$= w_1$

$T^2(v) = T(v)$ all v, $T = T^2$

$(\Longrightarrow)$ Let $T^2 = T$

wts. T is projection onto

$$W_1 = \{y \in V \mid T(y) = y\}$$

along $\ker(T)$

i. Wts. $V = W_1 \oplus \ker(T)$

a) $V = W_1 + \ker(T)$

For $v \in V$, write v as $\underbrace{w_1}_{\in W_1} + \underbrace{w_2}_{\in \ker(T)}$

Have $v \in V$, T

Want $w_1 \in W_1 = \{y \in V \mid T(y) = y\} \subseteq \operatorname{im}(T)$

and $w_2 \in \ker(T)$

$T(v) \in \operatorname{im}(T)$.

try showing this is in W_1

i.e. show $T(T(v)) = T(v)$

$T^2(v) = T(v)$

and $T^2 = T$.

$T(v) \in W_1$

$v \in V \rightsquigarrow \underline{T(v) \in W_1}$

wts $\underset{W_1}{v = ?} + \underset{\ker(T)}{?}$

$v = T(v) + \underset{\ker(T)}{?} \longrightarrow v - T(v)$

wts $v - T(v) \in \ker(T)$

$T(\underbrace{v - T(v)}_{\in \ker(T)}) = T(v) - T^2(v)$

$= T(v) - T(v) = 0$

$v = \underset{W_1}{T(v)} + \underset{\ker(T)}{(v - T(v))}$

Thus, $V = W_1 + \ker(T)$.

b) wts $W_1 \cap \ker(T) = \{0\}$.

Let $v \in W_1 \cap \ker(T)$.

wts $v = 0$.

$T(v) = v$ $\quad W_1$

and $T(v) = 0$ $\quad \ker(T)$

$\Downarrow$

$0 = T(v) = v$

Conclude, $\boxed{V = W_1 \oplus \ker(T).}$

ii. T is proj onto W_1
along W_2

$v = w_1 + w_2$ $\quad w_1 \in W_1$, $w_2 \in \ker(T)$

T is proj

wts $w_1 = T(v)$. $\quad W_1 \qquad \ker(T)$

in (i) showed $v = T(v) + (v - T(v))$

as $V = W_1 \oplus \ker(T)$ this is unique. $w_1 = T(v)$ by uniqueness.

$E_1(T)$ $\quad$ $\ker(T) = E_0(T)$

Remk. $W_1 = \mathrm{im}(T)$

$V = E_1(T) \oplus E_0(T)$

V fd VS $\quad T^2 = T$

$W_1 = \mathrm{im}(T)$ basis $\{v_1, \dots, v_k\}$

$\ker(T)$ basis $\{w_1, \dots, w_\ell\}$

$\beta = \{v_1, \dots, v_k, w_1, \dots, w_\ell\}$

k $\quad$ k+ℓ

$$[T]_\beta^\beta = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

k $\quad$ ℓ $\quad$ k+ℓ

$$\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

k $\quad$ ℓ

$[T]_\beta^\beta + (I - [T]_\beta^\beta) = I$

eval $\oplus$ v

$[T(v)]_\beta + [v - T(v)]_\beta = [v]_\beta$

$I - T$ proj onto $\ker(T)$ along $\mathrm{im}(T)$