Midterm @ 3PM

L 24 hrs

L 2.3 (matrix repr, coordinates)

deg $\leq n$ polys / $\mathbb{R}$ $\qquad$ $P(\mathbb{R})$

$$P_n(\mathbb{R}) \xrightarrow{D} \underline{P_n(\mathbb{R})} \quad \text{linear transformation}$$

$$p \longmapsto \frac{dp}{dx}$$

$$D(p + cq) = D(p) + cD(q)$$

$$(\operatorname{im}(D) \subseteq P_{n-1}(\mathbb{R}), \quad D: P_n(\mathbb{R}) \to \underline{P_{n-1}(\mathbb{R})})$$

[exc. $\quad \ker(D) = \underline{\text{constant fns}} \quad \dim 1$
$\quad \operatorname{im}(D) = P_{n-1}(\mathbb{R}) \quad \dim n \quad \Rightarrow n+1 = \dim P_n(\mathbb{R})$]

$$\mathcal{B} = \{1, x, \ldots, x^n\}$$

$$[1 + \underline{2}x + \underline{5}x^n]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ \vdots \\ 0 \\ 5 \end{pmatrix}$$

$$\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$$

$$\underline{\left([D]_{\mathcal{B}}^{\mathcal{B}}\right)_{ij} = a_{ij}}$$

$$D : P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$$

$$p \longmapsto p'$$

$v_j \in \mathcal{B}$

$\in \mathbb{R}$

$$D v_j = \sum_{i=0}^{n} a_{ij} v_i$$

$P_n(\mathbb{R}), \mathcal{B}$

$$D(1) = \overset{\text{0 polynomial}}{0} = \overset{0 \in \mathbb{R}}{\underline{0}} \cdot 1 + \overset{a_{21}}{\underline{0}} \cdot x + \overset{a_{31}}{\underline{0}} \cdot x^2 + \ldots + \overset{a_{n1}}{\underline{0}} \cdot x^n$$

$$[D(1)]_{\mathcal{B}} = \begin{array}{c} a_{11} \\ \\ \end{array} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftarrow \text{first col of } [D]_{\mathcal{B}}^{\mathcal{B}}$$

$$D(x) = \overset{\text{poly}}{1} = \overset{\mathbb{R}}{\underline{1}} \cdot \overset{\text{poly}}{1} + \overset{a_{22}}{\underline{0}} \cdot x + \overset{a_{32}}{\underline{0}} \cdot x^2 + \ldots + \overset{a_{n2}}{\underline{0}} \cdot x^n$$

a_{12}

$$[D(x)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + \ldots + 0 \cdot x^n$$

$$[D(x^2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$D(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2 + 0\cdot x^3 + \dots + 0\cdot x^n$

$$[D]_\beta^\beta = \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & & 0 \\ & 0 & 2 & 0 & & \vdots \\ & & 0 & 3 & \ddots & \\ & & & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & & n \\ & & & & & 0 \end{array}\right]$$

j^{th} col is $[D(x^{j-1})]_\beta$

j^{th} bas elt

$D(x^j) = jx^{j-1}$

$$[Dx^j]_\beta = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ j \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\right] j-1 \\ j^{th} \end{matrix}$$

$D(x^n) = nx^{n-1}$, $[D(x^n)]_\beta = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ n \\ 0 \end{pmatrix}$

$0\cdot 1 + 0\cdot x + \dots + 0\cdot x^{n-2} + n\cdot x^{n-1} + 0\cdot x^n$

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & 0 \\ & & & 3 & \\ & & \ddots & \ddots & \\ & 0 & & & n \\ & & & & 0 \end{bmatrix}$$

upper triangular

[$\deg(Dp) \le \deg(p)$]

$\beta = \{v_1, \ldots, v_n\}$ $\qquad \gamma = \{w_1, \ldots, w_m\}$

$T : V \longrightarrow \underline{W}$

$T(v_1)$ in the w_1 direction

$$\begin{array}{c|cccc} & T(v_1) & T(v_2) & \cdots & T(v_n) \\ \hline w_1 & \bullet & \bullet & & \\ w_2 & \bullet & \bullet & & \\ \vdots & & & & \\ w_m & & & & \end{array}$$

To find $[T]_\beta^\gamma$ $\qquad m \times n$

fill in column by column

1^{st} col, - $\underline{\text{evaluate}}$ $T(v_1)$

- $\underline{\text{expand}}$ $T(v_1)$ in basis γ

i.e. $T(v_1) = \underline{a_{11}}\, w_1 + \underline{a_{21}}\, w_2 + \cdots + \underline{a_{m1}}\, w_m$

i.e. $\underbrace{[T(v_1)]_\gamma}_{\text{first column of } [T]_\beta^\gamma} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$

2^{nd} col

3^{rd} col

$\vdots$

n^{th} col

$$\frac{x^j}{j}$$

$$\gamma = \left\{1, x, \frac{x^2}{2}, \frac{x^3}{3}, \ldots, \frac{x^n}{n}\right\}$$

$$\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$$

$$[D]_{\underline{\gamma}}^{\mathcal{B}} \leftarrow \text{dom}$$

$$D(1) = 0$$

$$D\underbrace{\left(\frac{x^j}{j}\right)}_{\gamma} = \frac{x^{j-1}}{1}, \quad [x^{j-1}]_{\underline{\mathcal{B}}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; j\text{th}$$

$$x^{j-1} = 0\cdot 1 + 0\cdot x + \ldots + 0\cdot x^{j-2} + 1\cdot x^{j-1} + 0\cdot x^j + \ldots + 0\cdot x^n$$

$$[D]_{\gamma}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ | & 0 & 1 & & | \\ | & | & 0 & \cdots & 0 \\ | & | & | & & 1 \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

Prop. V fd vs / F

$\beta = \{v_1, \ldots, v_n\}$ basis for V

$T : V \to V$.

$\rightarrow \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $A_{ij} = 0$, $i > j$

claim. i) $[T]_\beta^\beta$ upper triangular (matrix)

if and only if

ii) $T(v_j) \in \operatorname{span}\{v_1, \ldots, v_j\}$ for all j (lin trans.)

e.g. $D : P_n(\mathbb{R}) \to P_n(\mathbb{R})$

$\beta = \{1, x, \ldots, x^n\}$

$$[D]_\beta^\beta = \begin{array}{c|c} & 1 \quad x \quad x^2 \quad \cdots \quad x^n \\ \begin{array}{c} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{array} & \begin{pmatrix} 0 & 1 & & & \\ & \ddots & 2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & n \\ & & & & 0 \end{pmatrix} \end{array}$$

satisfies i)

$D(x^j) = j x^{j-1} \in \operatorname{span}\{1, x, \ldots, x^{j-1}, x^j\}$

satisfies ii)

in fact, $\in \operatorname{span}(1, x, \ldots, x^{j-1})$

$D(x^j)$ has no x^j term $\rightarrow$ diagonal entries are 0

$[T]_\beta^\beta$ upper Δ $\iff$ $T(v_j) \in \text{span}\{v_1, \dots, v_j\}$ for all j

pf.

Key ideas.

- $\begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} i^{th}$ $\iff$ no component past i $\iff$ $\text{span}(v_1, \dots, v_i)$

$(\Rightarrow)$. Let $[T]_\beta^\beta$ be upper triangular.

$$[T]_\beta^\beta = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

$$[T(v_1)]_\beta = [T]_\beta^\beta [v_1]_\beta$$
$$= [T]_\beta^\beta \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n$$
$$= \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$
[T]_\beta^\beta = \begin{array}{c|c}
 & \begin{array}{cccc} T(v_1) & T(v_2) & \cdots & T(v_n) \end{array} \\
\begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & 0 & a_{nn} \end{pmatrix}
\end{array}
$$

$$[T(v_1)]_\beta = \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$v_2, \dots, v_n$ have no contribution

$$T(v_1) = a_{11}v_1 + \boxed{\cancel{0v_2 + \cdots + 0v_n}}$$

$\in \operatorname{span}(v_1)$

$$[T(v_2)]_\beta = [T]_\beta^\beta [v_2]_\beta = [T]_\beta^\beta \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$[T(v_2)]_\beta = 2^{nd} \text{ col of } [T]_\beta^\beta = \begin{pmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$T(v_2) = a_{12}v_1 + a_{22}v_2 + \boxed{\cancel{0v_3 + \cdots + 0v_n}}$$

$T(v_2) \in \operatorname{span}(v_1, v_2)$

$$[T(v_j)]_\beta = [T]_\beta^\beta \, [v_j]_\beta$$

$$= [T]_\beta^\beta \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} j^{th} \text{ row}$$

$= j^{th}$ column of $[T]_\beta^\beta$

$$[T(v_j)]_\beta = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{jj} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$T(v_j) = a_{1j}\underline{v_1} + a_{2j}\underline{v_2} + \dots + a_{jj}\underline{v_j} + 0\cancel{v_{j+1} + \dots + 0 v_n}$$

$T(v_j) \in \mathrm{span}(v_1, \dots, v_j)$.

$(\Leftarrow)$ Let $T(v_j) \in \text{Span}(v_1, \dots, v_j)$ for all j

$$\text{Span}(v_1) \ni T(v_1) = a_{11}v_1 + 0 + \dots \quad \leftarrow \text{first col}$$

$$\text{Span}(v_1, v_2) \ni T(v_2) = a_{12}v_1 + a_{22}v_2 + 0 \dots \quad \text{2nd col}$$

$$\text{Span}(v_1, v_2, v_3) \ni T(v_3) = a_{13}v_1 + a_{23}v_2 + a_{33}v_3 + \dots \quad \text{3rd col}$$

$$\vdots$$

$$\text{Span}(v_1, \dots, v_j) \ni T(v_j) = a_{1j}v_1 + a_{2j}v_2 + \dots + a_{jj}v_j + \dots \quad j^{th} \text{ col}$$

$$[T]_\beta^\beta = \begin{pmatrix} a_{11} & a_{12} & a_{13} & & a_{1j} & & a_{1n} \\ 0 & a_{22} & a_{23} & & \vdots & & a_{2n} \\ \vdots & 0 & a_{33} & \dots & \vdots & & \vdots \\ \vdots & \vdots & 0 & & a_{jj} & \dots & \vdots \\ \vdots & \vdots & \vdots & & 0 & \ddots & \vdots \\ 0 & 0 & 0 & & 0 & & a_{nn} \end{pmatrix}$$

Thus, this is upper $\triangle$

$$\begin{array}{c|cc}
 & 1\ 2 & 3\ 4 \\
\hline
1,2 & * & * \\
3,4 & 0 & *
\end{array}$$

$T(E_{ij}) =$ some basis elt

$$M_{n\times n}(F) \xrightarrow{T} M_{n\times n}(F)$$

$$A \longmapsto A^t$$

$\beta = \{E_{11}, E_{12}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, E_{31}, \ldots, E_{3n}, \ldots, E_{n1}, \ldots, E_{nn}\}$

each col will have one 1 everything else is 0

$[T]_\beta^\beta$ $\quad n^2 \times n^2$

$$\begin{array}{cc}
E^{11} \cdots E^{1n} & E^{21} \quad E^{2n} \quad\quad E^{n1} - E^{nn}
\end{array}$$

$$\begin{pmatrix}
n\times n & n\times n & & \\
n\times n & n\times n & & \\
n\times n & n\times n & & \\
 & & & \\
 & & &
\end{pmatrix}$$