Basis

1. Span
2. linear independence

S spans V $S \subseteq V$

$\mathrm{span}(S) = V$

min'l subspace containing S

$\mathrm{span}(S) = \{ \overbrace{\sum a_i v_i}^{\in V} \mid a_i \in F, v_i \in S \} \subseteq V$

$S = \{v_1, v_2\}$, $\mathrm{span}(S) = \{ a_1 v_1 + a_2 v_2 \mid a_1, a_2 \in F \}$

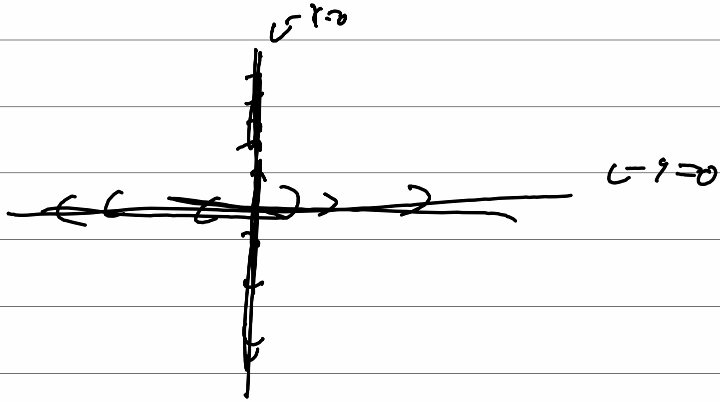
$\mathrm{span}(S) \subseteq V$ $v = \sum a_i v_i$, $a_i \in F$, $v_i \in S$

wts $V \subseteq \mathrm{span}(S)$

$v \in V \overset{\text{prove}}{\Longrightarrow} \boxed{v \in \mathrm{span}(S)}$

to show S spans V

take $v \in V$, write v as a lin combo of elts of S,



$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \underline{x=0} \text{ or } y=0 \right\} \subseteq \mathbb{R}^2$

$\begin{pmatrix} \pi \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -\sqrt{17} \end{pmatrix}$

$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

claim. $\operatorname{span}(S) = \mathbb{R}^2 \longrightarrow a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\underline{\in \operatorname{span}(S)}$

Pf. Let $\underline{u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}} \in \mathbb{R}^2$

want $\underline{a_1 s_1 + a_2 s_2} = u \qquad a_i \in \mathbb{R}, \; s_i \in S$

$\qquad \in S \quad \in S \qquad\qquad = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$P_2(\mathbb{R})$

$\{x^2+1, x, 1+x\}$ span $P_2(\mathbb{R})$

Let $p \in P_2(\mathbb{R})$

wts $p = a(x^2+1) + b(x) + c(1-x)$

$a, b, c \in \mathbb{R}$

$a_0 + a_1 x + a_2 x^2$

$\boxed{a = a_2, \quad b - c = a_1, \quad a + c = a_0}$

dim $V = n$ finite

$x + x^2 \in$ span

$S \subseteq V$ If $|S| = n$

Then S lin indep

$\Updownarrow$

S span V

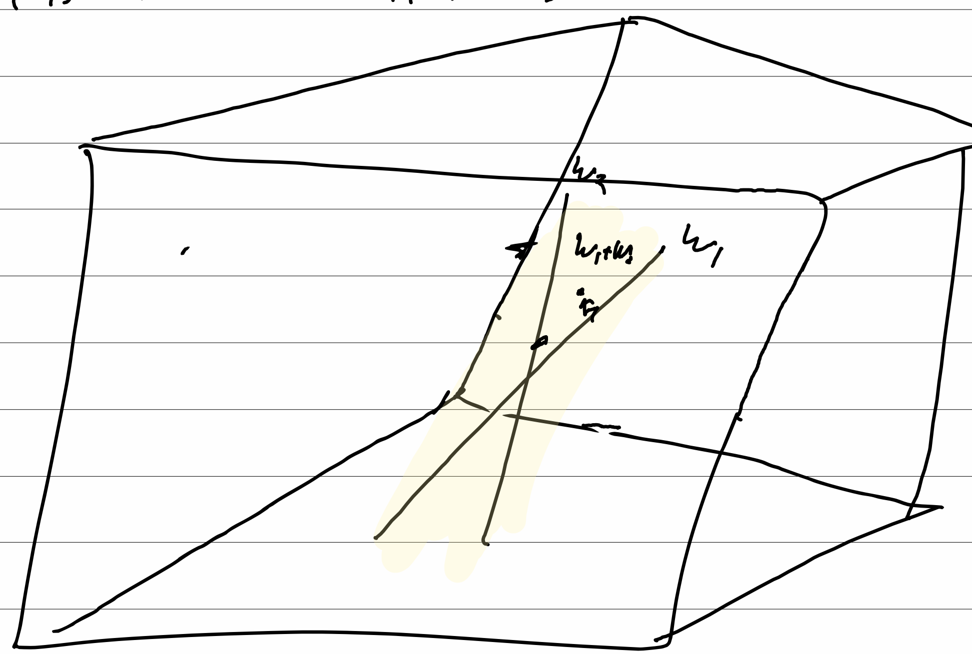
$\oplus$

V vector space

$W_1, W_2 \subseteq V$ subspaces.

$$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1,\ w_2 \in W_2 \}$$
$$= \mathrm{span}(W_1 \cup W_2) \ (\supseteq)$$
$$= \text{smallest subspace containing } W_1 \text{ and } W_2$$

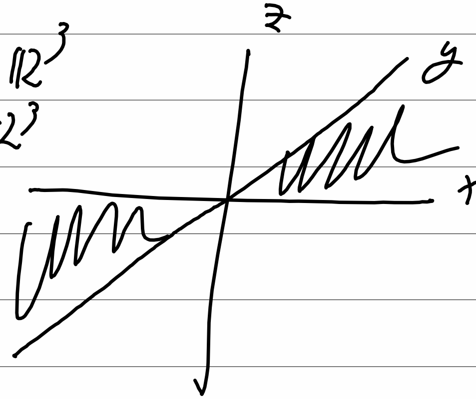
$$\mathrm{span}(S_1) + \mathrm{span}(S_2) = \mathrm{span}(S_1 \cup S_2)$$



$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$

$W_1 = x\text{-axis}$, in $\mathbb{R}^3$

$\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ $W_2 = y\text{-axis}$, in $\mathbb{R}^3$



$W_1 + W_2 = ?$

xy-plane

$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$W_1 \cap W_2 \neq \emptyset$

$0 + 0x + 0x^2 + \dots$

$W_1 \cap W_2 = \{0\}$

$a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots a_{2n}x^{2n}$

$P(\mathbb{R})$

$W_1 = \{ p \in P(\mathbb{R})$ s.t. only even power of x appear$\}$

$= \text{span}\{1, x^2, x^4, x^6, x^8, \dots\}$

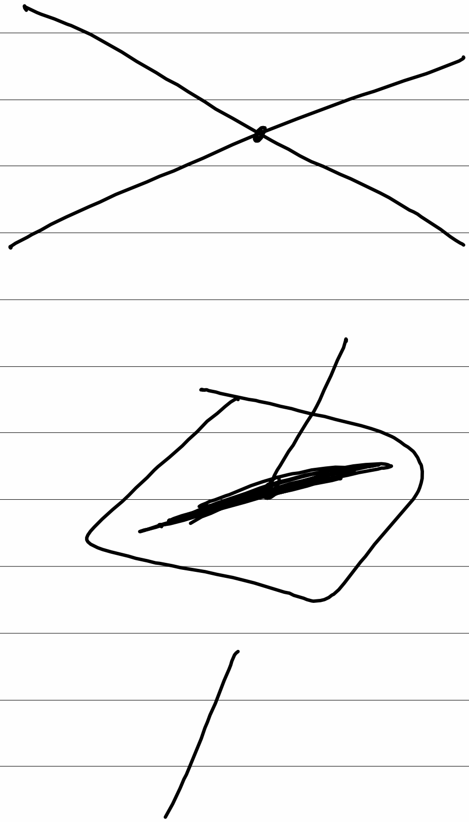
$W_2 = \{ p \in P(\mathbb{R})$ s.t. only odd power of x appear$\}$

$= \text{span}(x, x^3, x^5, x^7 \dots)$

$a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \dots$

$\in W_1 + \in W_2$

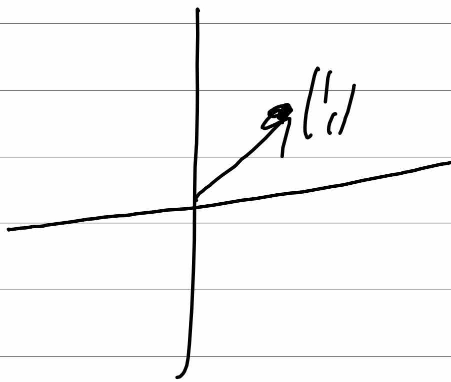
$W_1 + W_2 = P(\mathbb{R})$. $\quad p = \sum a_i x^i = \sum_{i\,\text{even}} a_i x^i + \sum_{i\,\text{odd}} a_i x^i$

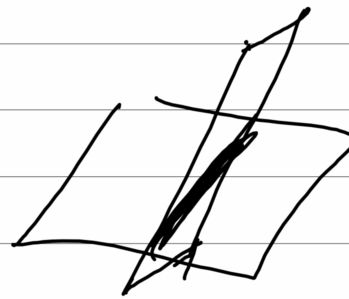


$\mathbb{R}^2$ $\quad + \leftarrow$ not usual addition

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{"}0\text{"} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$





$P(\mathbb{R})$

$\underline{V} = \underline{W_1 + W_2}$ $\iff$ for all $v \in V$, there are $w_i \in W_i$ s.t. $v = w_1 + w_2$

$W_1 \cap W_2 = \{0\}$, write as $V = W_1 \oplus W_2 \leftarrow$ direct sum

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\in W_1} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\in W_2} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\in W_1} - \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\in W_2}$$

$v = w_1 + w_2$ uniquely

$= w_1' + w_2' \implies w_1 = w_1'$, $w_2 = w_2'$

$W_1 + W_2 = \mathbb{R}^3$

$W_1 \cap W_2 = y\text{-axis} \neq \{0\}$.

$W_1 = xy$ plane in $\mathbb{R}^3$, $W_2 = yz$ plane in $\mathbb{R}^3$

$V = W_1 + \dots + W_k$

$\Leftrightarrow$

for all $v \in V$, s.t. $v = w_1 + \dots + w_k$

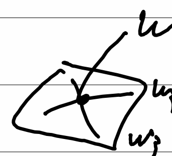
exist $w_i \in W_i$

$V = W_1 \oplus \dots \oplus W_k$

$\Leftrightarrow$

for all $v \in V$, there is unique $w_i \in W_i$ s.t.

$v = w_1 + \dots + w_k$

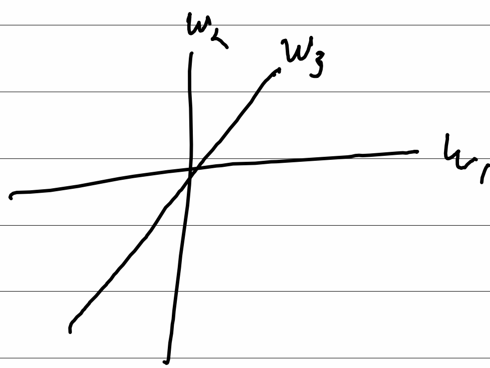


$\Leftrightarrow$ $V = W_1 + \dots + W_k$

and

for all i $\underbrace{W_i} \cap \underbrace{\left(\sum_{j \neq i} W_j \right)} = \{0\}$.

$W_1 \cap (W_2 + W_3)$
$\Downarrow$
$\{0\}$



$W_1 \cap (W_2 + W_3)$

$= W_1 \cap (\mathbb{R}^2)$

$= W_1$

$\neq \{0\}$.

$V = \{f : \mathbb{R} \longrightarrow \mathbb{R}\}$

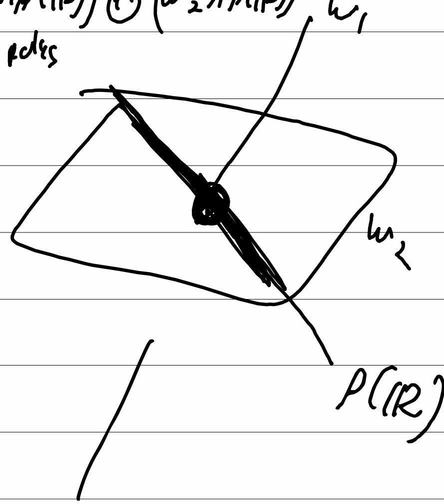
$W_1 = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = f(-x)\}$ even

$W_2 = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = -f(x)\}$ odd

exc. $\underbrace{V = W_1 \oplus W_2}_{U}$

odd poly's $\qquad V$ drawn as $\mathbb{R}^3$

$P(\mathbb{R}) = \underbrace{(W_1 \cap P(\mathbb{R}))}_{\text{even poly's}} \oplus (W_2 \cap P(\mathbb{R}))$



$S = \{v_1, \ldots, v_k\}$

S lin indep iff $\quad \mathrm{span}(S) = \mathrm{span}(v_1) \oplus \ldots \oplus \mathrm{span}(v_k)$

fact. $S = \{v_1, \dots, v_k\}$

ex. $\text{span}(S) = \text{span}(v_1) + \text{span}(v_2) + \dots + \text{span}(v_k)$

pf. Let $v \in \text{span}(S)$

Then $v = \underbrace{a_1 v_1}_{\text{span}(v_1)} + \dots + \underbrace{a_k v_k}_{\text{span}(v_k)}$, $a_i \in F$.

$\therefore v \in \text{span}(v_1) + \dots + \text{span}(v_k)$

$\text{span}(S) \subseteq \text{span}(v_1) + \dots + \text{span}(v_k)$

Let $v \in \text{span}(v_1) + \dots + \text{span}(v_k)$.

$v = w_1 + \dots + w_k$, $w_i \in \text{span}(v_i)$.

$w_i = a_i v_i$

$v = \sum a_i v_i \in \text{span}(S)$

$\text{span}(v_i) \subseteq \text{span}(S)$, $\{v_i\} \subseteq S$

$\therefore \sum \text{span}(v_i) \subseteq \text{span}(S)$

fact. $S = \{v_1, \dots, v_k\}$

$$\text{span}(S) = \text{span}(v_1) + \dots + \text{span}(v_k)$$

from befor

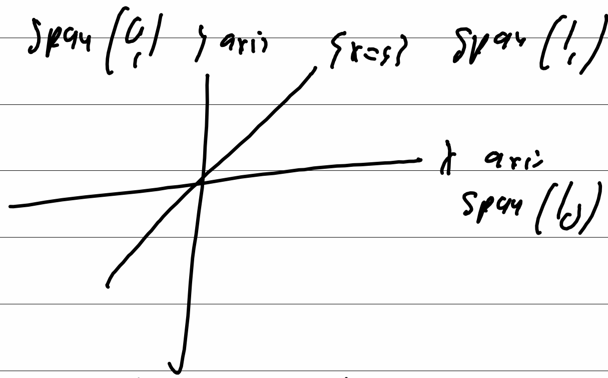
Furthermore,

exc. (hard)

S lin indep

$\Updownarrow$

$$\text{span}(S) = \text{span}(v_1) \oplus \dots \oplus \text{span}(v_k)$$



$S = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$

not lin indep

$\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} + \text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\} + \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ not a direct sum

$T : V \longrightarrow W$ linear

[a) $V_1 \subseteq V$ subspace

$T(V_1) = \{T(v) \mid v \in V_1\}$

subspace of W

b) $W_1 \subseteq W$ subspace

$(T^{-1}(W_1) =) \{v \in V \mid T(v) \in W_1\}$ is a subspace of V

pf. a). goal. $T(V_1) \subseteq W$ subspace.

$0 \in T(V_1)$

$T(V_1)$ closed under $+$

$T(V_1)$ closed under scalar mult } inherited from W

$a \cdot_{F,W} (b \cdot_{F,W} w) = (a \cdot_F b) \cdot_{F,W} w$

$a \cdot (bw) = (ab) w$

$(a +_F b) \cdot_{F,W} w = a \cdot_{F,W} w +_W b \cdot_{F,W} w$

$(a+b) w = aw + bw$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad T(x) = x+1$

$\{0\} \longmapsto \cancel{\{1\}}$

$T(V_1) \subseteq W$ is a subspace

i. $0 \in T(V_1)$

ii. $T(V_1)$ closed under $+$

iii. $T(V_1)$ closed under scalar mult

ii. $\alpha, \beta \in T(V_1) \Rightarrow \alpha+\beta \in T(V_1)$

iii. $\alpha \in T(V_1),\ c \in F \Rightarrow c\alpha \in T(V_1)$

$\alpha, \beta \in T(V_1)$ and $c \in F$
then $\alpha + c\beta \in T(V_1)$

ii. $\alpha, \beta \in T(V_1)$.

wts $\alpha + \beta \in T(V_1)$

$\alpha = T(v)$ some $v \in V_1$

$\beta = T(v')$ some $v' \in V_1$

$$\alpha+\beta = T(v) + T(v') \overset{T \text{ linear}}{=} T(v+v')$$

wts $T(v+v') \in T(V_1)$

indeed, $v+v' \in V_1$ as $v, v' \in V_1$ and $V_1 \subseteq V$ subspace. Thus, $\alpha+\beta \in T(V_1)$.

iii. $c \in F$, $\alpha \in T(U_1)$

wts $c\alpha \in T(U_1)$

$\alpha = T(v)$ some $v \in U_1$

$c\alpha = cT(v)$

$= T(cv)$ as T is linear

$cv \in U_1$ as U_1 is a subspace, $v \in U_1$, $c \in F$

$c\alpha \in T(U_1)$

i. wts $0 \in T(U_1)$

$T(0) = 0$, $0 \in U_1$.

, $W_1 \subseteq W$ subspace

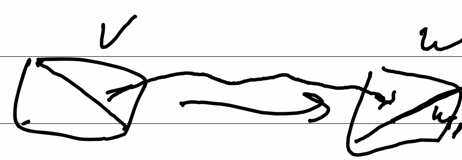
, $T: V \to W$

b). $\{v \in V \mid \boxed{T(v) \in W_1}\} \subseteq V$ subspace.

$v, v' \in U$, and $c \in F$

wts $v + cv' \in U$

$T(v), T(v') \in W_1$



wts $T(v + cv') \in W_1$

$$\|$$

$$\underbrace{T(v)}_{W_1} + c\,\underbrace{T(v')}_{W_1} \in W_1$$

$T(0) = 0 \therefore 0 \in$ set

$v \qquad w \qquad v$

$$T(v+cv') = T(v)+T(cv')$$
$$= T(v) + cT(v')$$

$w, \quad w'$

1+1 1+1 1+1+1+1+1+1+1

$$3x+2y+7z=0$$
$$x+z=0$$
$$y+2z=0$$

$$\begin{pmatrix} 3 & 2 & 7 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$x+z=0 \qquad x=-z$

$y+2z=0 \qquad y=-2z$

a basis for the set of solutions

basis for kernel of matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -2z \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad x = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x \\ z \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

basis for sol set