Midterm Th 3

(OH)

Review session Wed

finite dimensional

V a vector space.

$\mathcal{B}$ is a basis for V

$\mathcal{B} = \{v_1, \dots, v_n\}$ lin indep

spans

$v \in V$, $v = a_1 v_1 + \dots + a_n v_n$ $a_i \in F$ ($\mathcal{B}$ spans)

The a_i are unique ($\mathcal{B}$ lin indep)

$$V \xrightarrow{\text{invertible}} F^n$$

$$v \longmapsto \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

$$\sum a_i v_i \longleftarrow\!\shortmid$$

$v = a_1 v_1 + \dots + a_n v_n$

Notation. $[v]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \iff v = a_1 v_1 + \dots + a_n v_n$

$P_3(\mathbb{R})$ ordered basis $\{1, x, x^2, x^3\}$

$\Downarrow$

$\underset{0x\ 0x^2}{1+x^3} \longleftrightarrow \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \in \mathbb{R}^4$

Fact. $T: V \longrightarrow W$ <u>linear</u>

$\mathcal{B}$ basis for V

T is "uniquely determined by its action on $\mathcal{B}$"

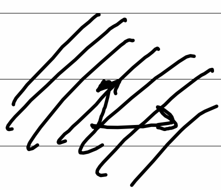
$U: V \longrightarrow W$

s.t. $T(v) = U(v)$ for all $v \in \mathcal{B}$

$\Rightarrow T = U$

<u>$\mathcal{B} = \{v_1, \dots, v_n\}$</u>. $T(v_i) = U(v_i)$ for all i

$\Downarrow$



$T = U$ ($T(v) = U(v)$ for all $v \in V$)

$T: V \longrightarrow W$

β ordered basis for V $\qquad$ γ ordered basis for W

$\{v_1, \dots, v_n\}$ $\qquad$ $\{w_1, \dots, w_m\}$

$T(v_j) \in W$

$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$

T is "determined" by these field elements

$T \longleftrightarrow (a_{ij})_{ij}$

$[T]_\beta^\gamma$ matrix whose entries are a_{ij}

$m \times n$ matrix

$([T]_\beta^\gamma)_{ij} = a_{ij}$

alt. notation $\quad {}_\gamma[T]_\beta$

fact. $T: V \longrightarrow W$

β $\quad\quad$ γ

n elts $\quad$ m elts

exr. 2.3.1

$$\in \underbrace{[T(v)]_\gamma}_{\in F^m} = \underbrace{[T]_\beta^\gamma}_{\in M_{m\times n}(F)} \underbrace{[v]_\beta}_{\in F^n}$$

coord rep of $T(v) \in W$ in ordered basis γ

matrix repr of T in the two ordered bases: β, γ

coord repr of $v \in V$ in ordered basis β

$$\left([T(v)]_\gamma = \underline{{}_\gamma[T]_\beta} \, [v]_\beta \right)$$

$$v \longmapsto T(v)$$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\beta"'] & W \arrow[d, "\gamma"] \\
F^n \arrow[r, "{[T]_\beta^\gamma}"'] & F^m
\end{tikzcd}$$

$$[v]_\beta \longmapsto [T]_\beta^\gamma [v]_\beta$$

$$[T(v)]_\gamma \;\; \shortparallel$$

$\sum a_i v_i = \sum b_i v_i$
$\Downarrow$
$a_i = b_i$ for all i

$\beta = \{v_1, \ldots, v_n\}$ lin indep

$\gamma = \{w_1, \ldots, w_m\}$

$$[T(v_i)]_\gamma = [T]_\beta^\gamma [v_i]_\beta$$

$[v_i]_\beta \in F^n$

? $\parallel$

$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$$v_1 = \sum_{j=1}^{n} c_j v_j$$

lin combo of basis elts

$[v_i]_\beta = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ position

$$v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$$

$c_1 = 1, \quad c_2 = 0, \ldots, \quad c_n = 0$

$[v_1]_\beta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{matrix}$

$$v_i = 0 \cdot v_1 + \cdots + 1 \cdot v_i + \cdots + 0 \cdot v_n$$

$$[T(v_i)]_\gamma = [T]_\beta^\gamma \underbrace{[v_i]_\beta}_{\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}}$$

$0 - 1 - 0$

i^{th} col of $[T]_\beta^\gamma$

Fact: $\underbrace{i^{th} \text{ col of } [T]_\beta^\gamma \text{ is } [T(v_i)]_\gamma}_{\text{rephrasing def of } [T]_\beta^\gamma}$

$$F^3 \xrightarrow{T} F^2$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} a+b \\ b+c \end{pmatrix}$$

$\beta = \{e_1, e_2, e_3\}$ F^3

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\gamma = \{e_1, e_2\}$ F^2

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$[T]_\beta^\gamma$

$e_1 \in \beta$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\longmapsto T(e_1) = \begin{pmatrix} 1+0 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot e_1 + 0 \cdot e_2$

$e_2 \longmapsto T(e_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot e_1 + 1 \cdot e_2$

$e_3 \longmapsto T(e_3) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \cdot e_1 + 1 \cdot e_2$

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\gamma = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \right\} \text{ basis for } F^2$$

$$[T]_\beta^\gamma$$

$$e_1 \longmapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \cdot v_1 + 1 \cdot v_2 \overset{\gamma}{\longleftrightarrow} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$e_2 \longmapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot v_1 + 0 \cdot v_2 \overset{\gamma}{\longleftrightarrow} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$e_3 \longmapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot v_1 - 1 \cdot v_2 \overset{\gamma}{\longleftrightarrow} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$M_{2\times2}(F) \xrightarrow{\quad T \quad} M_{2\times2}(F) \qquad \text{linear}$$

$$A \longmapsto A^t$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\underline{\beta} = \left\{ \overset{E_{11}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underbrace{\overset{E_{12}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}}_{2^{nd}}, \underbrace{\overset{E_{21}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}}_{3^{rd}}, \underbrace{\overset{E_{22}}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}}_{4^{th}} \right\}$$

basis for $M_{2\times2}(F)$.

$$[T]_\beta^\beta$$

$$E_{11} \longmapsto E_{11}^t = \underline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} = 1 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{22} \xrightarrow{\beta} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E_{12} \longmapsto E_{12}^t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \underline{E_{21}} \overset{\beta}{\longleftrightarrow} \underline{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{3^{rd}}$$

$$E_{21} \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E_{12} \overset{\beta}{\longleftrightarrow} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow 2^{nd}$$

$$E_{22} \longmapsto \begin{pmatrix} 0 & c \\ 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix} \overset{\beta}{\longleftrightarrow} \begin{pmatrix} 0 \\ a \\ c \\ 1 \end{pmatrix} \in \mathbb{R}^4$$

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

tells you

$E_{21} \longmapsto 0 \cdot E_{11} + 1 \cdot E_{12}$
$+ 0 \cdot E_{21} + 0 \cdot E_{22}$

$\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$\underline{P_2(\mathbb{R})} \xrightarrow{T} \underline{M_{2\times 2}(F)} \quad \text{linear}$$

$\beta = \{1, x, x^2\}$

$$p = a_0 + a_1 x + a_2 x^2 \longmapsto a_0 I + a_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2$$

$I = E_{11} + E_{22}$

$$1 \longmapsto I \qquad [I]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$x \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = E_{11} + E_{12} + E_{22}$$

$$[T(x)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$x^2 \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = E_{11} + 2E_{12} + E_{22}$$

$$[T(x^2)]_\gamma = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$[T]_\beta^\gamma = \begin{array}{c c} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{array}{c} E_{11} \\ E_{12} \\ E_{21} \\ E_{22} \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$