OH today 3PM

Midterm next Th

extra OH/ review session next week

"char $\neq 2$" 1, 5, 13

$\mathbb{F}_2 = \{0, 1\}$

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

"characteristic 2"

"$2 = 0$" $1+1=0$

$2 = 0$

$\mathbb{F}_2$ char 2

5   7

$5 = 0$ $7 = 0$

$\mathbb{R}$: 0: 0, 1, 2, ... distinct

F be a field.

Consider $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \in F^3$

Lin indep/dep, $\underbrace{\text{over } F}$

a) $F = \mathbb{R}$, $\mathbb{R}^3$

$$\underline{a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}}$$

$$\Updownarrow$$

$$\left(\begin{array}{ccc|c}1&1&0&0\\1&0&1&0\\0&1&1&0\end{array}\right)$$

unique solution $\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix} \sim \begin{pmatrix}1&1&0\\0&-1&1\\0&1&1\end{pmatrix} \sim \begin{pmatrix}1&1&0\\0&1&-1\\0&1&1\end{pmatrix}$$

$$\sim \begin{pmatrix}1&0&1\\0&1&-1\\0&0&2\end{pmatrix} \overset{\div 2}{\sim} \begin{pmatrix}1&0&1\\0&1&-1\\0&0&1\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

lin indep in $\mathbb{R}^3$

b) $F = \mathbb{Z}_2$ $\qquad (\mathbb{Z}_2)^3$

$$a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 0\\1 & 0 & 0\\0 & 1 & 1\end{pmatrix} \overset{\dots}{\sim} \begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{pmatrix}$$

$1+1$ in F

$2=0$ in $\mathbb{Z}_2$

$$\begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{pmatrix}$$

lin dependency

$a + c = 0$

$b - c = 0$

c free

$$\begin{pmatrix}-c\\c\\c\end{pmatrix} \qquad \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

characteristic 2 fields

$+, \cdot$

1

$\cancel{0} \in F$

$1+1=0 \Rightarrow 1=-1 \Rightarrow a=-a$

$a \in F$

$2=0$

$\cancel{\frac{A}{2}}$

$(a+b)(a+b) = a\cdot a + b\cdot b$

F

$(a+b)^0 \quad F \quad (a+b)^2 = a^2 + \underbrace{2ab}_{2=0} + b^2$

$= a^2+b^2$ true in characteristic 2

$\mathbb{R}$

characteristic $\neq 2$

$\ast\ast\ast$

$1+1\neq 0$

$2\neq 0$

$\frac{1}{2}$ exists

$a \neq -a$ if $a\neq 0$

$\Rightarrow \mathbb{R}, \mathbb{Q}, \mathbb{C}$

Linear transformations.

Let V, W vector spaces over a field F

Let $T : V \longrightarrow W$ function.

We say T is linear (linear transformation) if

for all v, v' in V
$$T(v + v') = T(v) + T(v')$$

for all v in V, $a \in F$
$$T(av) = aT(v)$$

Communicate between V, W

$$T\left(\sum_{i=1}^{k} a_i v_i\right) = \sum_{i=1}^{k} a_i T(v_i)$$

"distributes over linear combinations"

$\{x^3-x,\ 2x^2+4,\ -2x^3+3x^2+2x+6\}$ in $P(\mathbb{R})$

$$a(x^3-x)+b(2x^2+4)+c(-2x^3+3x^2+2x+6)=0$$

$$\vdots$$

$$\underbrace{(a-2c)}_{=0}\underbrace{x^3}_{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}+\underbrace{(2b+3c)}_{=0}\underbrace{x^2}_{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}+\underbrace{(-a+2c)}_{=0}\underbrace{x}_{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}+\underbrace{(4b+6c)}_{=0}\underbrace{1}_{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}=0$$

4 equations

$$\begin{array}{ccc} & T(2x^2+4) & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 4 & 6 & 0 \end{array}\right)$$
$$T(x^3-x) \qquad T(-2x^3+3x^2+2x+6)$$

$$T : P(\mathbb{R}) \longrightarrow \mathbb{R}^4$$

$$p=\sum_{i=0}^{k} a_i x^i,\quad T(p)=\begin{pmatrix}a_0\\a_1\\a_2\\a_3\end{pmatrix}$$

- T linear. $T(p+q)=T(p)+T(q)$,

Let $p(x)=\sum_{i=0}^{k} a_i x^i$, $q(x)=\sum_{i=0}^{l} b_i x^i$

$$p+q=\sum_{i=0}^{\max(k,l)}(a_i+b_i)x^i$$

$$T(p+q) = \begin{pmatrix} a_0+b_0 \\ a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{T(p)} + \underbrace{\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{T(q)}$$

$$= T(p) + T(q)$$

Let $\lambda \in \mathbb{R}$, $p \in P(\mathbb{R})$. Let $p = \sum a_i x^i$

Then $\lambda p = \sum \lambda a_i x^i$

$$T(\lambda p) = \begin{pmatrix} \lambda a_0 \\ \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix} = \lambda \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \lambda T(p)$$

So T is <u>linear</u>.

$T: P(\mathbb{R}) \to \mathbb{R}^4, \quad p \mapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$

$$\ker(T) = \{ p \in P(\mathbb{R}) \mid T(p) = \overset{0 \text{ in } W = \mathbb{R}^4}{0} \}$$

Let $T(p) = 0$. If $p = \sum_{i=0}^{k} a_i x^i$, then

$$T(p) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

for $T(p) = 0$, we therefore have

$$T(p) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

equivalent

$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0$

$\cancel{1+x} \qquad \cancel{2+x^4}$

basis $\{x^4, x^5, x^6, \dots\}$

$$\ker(T) = \{ p \in P(\mathbb{R}) \mid p = a_4 x^4 + a_5 x^5 + \dots \}$$

(x^4 is a factor of p)

subspace of $P(\mathbb{R})$

$$\mathrm{im}(T) = \{ w \in \mathbb{R}^4 \mid \text{there is } p \in P(\mathbb{R}) \text{ w/ } T(p) = w \}$$

$$= \{ T(p) \mid p \in P(\mathbb{R}) \}$$

= outputs of T

subspace of $\mathbb{R}^4$

$$T(p) = \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}}$$

$p = \sum a_i x^i$

Let $w \in \mathbb{R}^4$. $w = \left[\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} \right]$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

basis for $\mathrm{im}(T)$

Can we find $p \in P(\mathbb{R})$ s.t. $T(p) = w$?

$$p = w_0 + w_1 x + w_2 x^2 + w_3 x^3 \in P(\mathbb{R})$$

$$T(p) = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} = w$$, so w in $\mathrm{im}(T)$

Thus, $\mathrm{im}(T) = \mathbb{R}^4$

onto, surjective

Lemma. $T: V \to W$ linear. If $v_1, \dots, v_k$ in V s.t. $T(v_1), \dots, T(v_k)$ lin indep (in W). Then $v_1, \dots, v_k$ lin indep (in V)

Pf. Let $\underbrace{\sum a_i v_i = 0}$ (in V)

$a_i \in F$, $v_i \in V$, $0 \in V$

$$T\left(\sum a_i v_i\right) = T(0)$$

$$\sum a_i T(v_i) = 0 \quad \text{in } W$$

$\{T(v_i)\}$ lin indep

Thus, $a_i = 0$ for all i

So $v_1, \dots, v_k$ lin indep. $\square$

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are $x^3 - x$, $2x^3 + 4$, $-2x^3 + 3x^2 + 2x + 6$ lin indep?

If $T(x^3 - x), \dots, \dots$ are lin indep in $\mathbb{R}^4$ then they are lin indep in $P(\mathbb{R})$

$C^\infty(\mathbb{R})$

$\rightsquigarrow$

smooth

$V = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is infinitely differentiable}\}$

$$V \xrightarrow{D} V$$

$$f \longmapsto \frac{df}{dx}$$

$\ker(D) = \{f \mid \frac{df}{dx} = 0\}$ = constant fns

$\operatorname{im}(D) = V$ by integration

$$V \xrightarrow{D - I} V$$

$$f \longmapsto \frac{df}{dx} - f$$

$\ker(D - I) = \{f \mid \frac{df}{dx} - f = 0\}$

$\frac{df}{dx} = f$, , $f(x) = \lambda e^x$

$= \operatorname{Span}(\underbrace{e^x}_{\in V})$

$\overset{\text{subspace of } V}{W} \subseteq \text{span}(W)$

$\overset{\text{subset of } V}{S} \subseteq \text{Span}(S)$

Let $s \in S$. wts $s \in \text{span}(S)$

$$\text{Span}(S) = \bigcap_{\substack{S \subseteq W \leq V \\ \text{subspace}}} W$$

wts $s \in$

That is, we want to show that for all subspaces $W \leq V$ s.t. $S \subseteq W$, that $s \in W$.

Why is $s \in W$?. By assumption, $s \in S$.

" " $S \subseteq W$

Thus $s \in W$.

So $s \in \bigcap_{\substack{S \subseteq W \leq V \\ \text{subspace}}} W = \text{Span}(S)$

Thus, $S \subseteq \text{Span}(S)$

$W = \mathrm{span}(W)$ prop 7.1.

Let W be a subspace of U

. wts $W = \mathrm{span}(W)$.

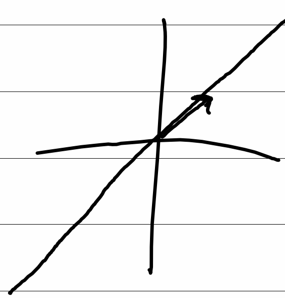
sts that W satisfies the property characterizing $\mathrm{span}(W)$, i.e. being the minimal subspace containing W prop 7.1

- W is a subspace by hypothesis

- W contains W. $W \subseteq W$

$w \in W$. Then $w \in W$. so $W \subseteq W$.

- minimal.



$L \subseteq W_1$

$L \subseteq W_2$

min'l subspace satisfying a property P means W satisfies P and if W' satisfies P then $W \subseteq W'$

wts. If W' is a subspace containing w, then $W \subseteq W'$

$\underbrace{\text{containing } w}_{W \subseteq W'}$

Let $w \in W$. wts $w \in W'$
$W \subseteq W'$ by hypothesis

Thus, W is the minimal subspace of V containing w

So (by prop 7.1), $W = \text{span}(w)$.

1.3.28

- $0 \in W$
- $c(x+y) \in W \quad c \in F, \; x, y \in W$

$c=1 \Rightarrow$ closed under $+$

$y=0 \Rightarrow$ closed under scalar mult

- $x+y \in W \quad x, y \in W$
- $cx \in W \quad c \in F, \; x \in W$

$x+(-1)\cdot x \in W$

$0 \in W$

- $x+cy \in W \quad x, y \in W, \; c \in F$

(exc. that this axiom $\Leftrightarrow$ W subspace)

$$(c\cdot(x+y))_{ij} = c\,(x+y)_{ij} = c(x_{ij}+y_{ij}) \quad x, y \in W$$
$$= c(-x_{ji} - y_{ji})$$
$$= -c\,(x_{ji} + y_{ji})$$
$$= -c\,(x+y)^{*}_{ij}$$

$T: M_n(F) \longrightarrow M_n(F)$ linear

$A \longmapsto \underbrace{A}_{I} + \underbrace{A^t}_{t}$

nullspace

$\ker(T)$?

$= \{A \in M_n(F) \mid T(A) = 0\}$

a matrix

$\{A \in M_n(F) \mid A + A^t = 0\}$

$\Updownarrow$

$A = -A^t$

A skew-symmetric

$T(A) = 0 \iff A = -A^t$ linear in A

$\iff A$ skew symmetric

$\ker(T) = W$ $\therefore$ W is the ker of a lin trans, thus a subspace

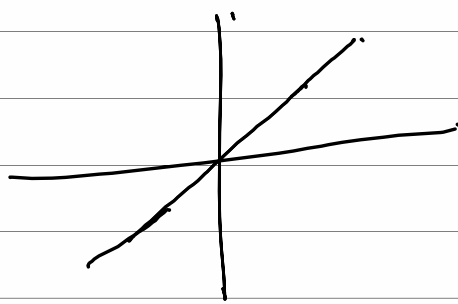
$T(X) = 0$

$W_1 \oplus W_2 = M_n(F)$

- $W_1 \cap W_2 = \{0\}$

- $W_1 + W_2 = M_n(f)$

Every $A \in M_n(f)$ is written (uniquely) as
$A_1 + A_2$, $A_1 \in W_1$
$A_2 \in W_2$



W_1 = Skew sym
W_2 = sym

$A = (\text{Skew sym}) + (\text{sym})$
$= \frac{A - A^*}{2} + \frac{A + A^*}{2}$ $(2 \neq 0)$

$V = W_1 + W_2$

$V = W_1 \oplus W_2$

$$A = \frac{(A - A^t)}{2} + \frac{(A + A^t)}{2}$$

$V = W_1 + W_2$

$V \longrightarrow W_1$

given a matrix A, find a skew sym matrix

$A \rightsquigarrow B$ s.t. $B^t = -B$

A, A^t 22,

$B = \underline{A - A^t} \longrightarrow \frac{1}{3}A - \frac{1}{3}A^t$ $\lambda A - \lambda A^t$

$A \rightsquigarrow \underbrace{A - A^t}_{W_1}$ skew sym

$A \rightsquigarrow \underbrace{A + A^t}_{W_2}$ sym

$\frac{1}{2}$ char $\neq 2$

$$\frac{(A - A^t) + (A + A^t) = 2A}{2}$$

$$A \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} (A + A^{t}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$V = \{ \text{fn } f : \mathbb{R}^2 \to \mathbb{R} \}$

$W_1 =$ even fn $\quad f(x) = f(-x)$

$W_2 =$ odd fns $\quad f(x) = -f(x)$

$V = W_1 \oplus W_2$ similarly

$M_n(F) = W_1 \oplus W_2$

$-\ W_1 + W_2 = M_n(F)$ ✓

$-$ wts $W_1 \cap W_2 = \{0\}$.

$\{0\} \subseteq W_1 \cap W_2$ ✓

wts $W_1 \cap W_2 \subseteq \{0\}$

$\subseteq M_n(F)$

Let $w \in W_1 \cap W_2$. Show $w = 0$.

$w \in W_1$ and $w \in W_2$

w skew-sym and w sym

$w^t = -w$ and $w^t = w$

Thus, $-w = w$ char $\neq 2 \Rightarrow w = 0$

$W_1 = W_2$

$\frac{1}{2}$, $1 = -1$

$2w = 0$, $2 \neq 0$ as char $\neq 2$

thus $\frac{1}{2} \cdot 2w = \frac{1}{2} \cdot 0$

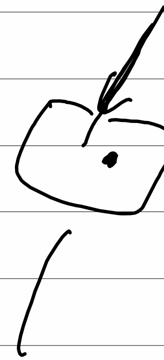
$w = 0$

Sym $\qquad\qquad\qquad$ skew sym $\qquad\qquad$ $A \in W_1 \cap W_2$

$$A = \frac{A+A^T}{2} = \frac{A-A^T}{2} = A$$

$W_2 \qquad\qquad W_1$

$$M_n(F) = W_1 + W_2$$



Sym

$$w \in W_1 \cap W_2$$

$$A \longmapsto \frac{A-A^T}{2}$$

$$\begin{tikzcd}
M_n(F) \arrow[r] \arrow[d] & W_1 \\
W_2 & W_1 \cap W_2 \arrow[l] \arrow[u]
\end{tikzcd}$$

$$A \longmapsto \frac{A+A^T}{2}$$

$$\frac{A+A^T}{2} = \frac{B-B^T}{2}$$

$V = W_1 \oplus W_2$

$V \xrightarrow{P} W_1 \hookrightarrow V$

$P(P(v)) = v$

$$\frac{\left(\frac{A+A^*}{2}\right) + \left(\frac{A+A^*}{2}\right)^*}{2} = \frac{A+A^*}{2}$$

S sym, $\frac{S+S^*}{2} = S$

16.12. $\{-A = A^t\}$ subspace ✓

$-\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ $\overset{\mathbb{F}_2}{=}$

basis

dim

$\begin{pmatrix}1&0\\0&1\end{pmatrix}^t = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$n=2$ $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ 1 dim

$\boxed{(\text{? } \cancel{\text{dim}} \text{ char } 2)}$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^t = -\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$$\begin{pmatrix}a&c\\b&d\end{pmatrix} = \begin{pmatrix}-a&-b\\-c&-d\end{pmatrix}$$

$a = -a \Rightarrow a = 0$

$c = -b$

$b = -c$

$d = -d \Rightarrow d = 0$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}0&b\\-b&0\end{pmatrix}$$

all $\text{SkewSym}_2(F)$

$b\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} =$$

$$\left.\begin{aligned} a &= -a \\ e &= -e \\ i &= -i \end{aligned}\right\} 0$$

$$\begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix} \longrightarrow 3 \text{ \#'s to describe this}$$

$$= \underline{b} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \underline{c} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$+ \underline{f} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$i<j$ → freely choose

$$\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & & \vdots \\ \vdots & & \ddots & \\ -a_{1n} & \cdots & & 0 \end{pmatrix}^t = -A$$

$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$ for all i

$a_{ij} = -a_{ji}$

$$\sum_{i<j} a_{ij} \begin{array}{c c} & ij \\ ji & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array}$$

$\underbrace{S^{ij}}_{i<j}$ = matrix with ij entry 1, ji entry -1, all else 0

$\sum_{i<j} a_{ij} S^{ij}$ basis

→ span

lin indep

$$\underbrace{\sum \underbrace{a_{ij}}\, \delta^{ij}} = 0$$

$$\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ \text{above} & & & a_{n-1,n} \\ & & & 0 \end{pmatrix} = 0$$

$\therefore$ $a_{ij} = 0$ always

$\{\underline{\delta^{ij}} \mid i<j\}$ basis

$i=1$	δ^{12}	δ^{13} ...	δ^{1n}	$n-1$
$i=2$		δ^{23} ...	δ^{2n}	$n-2$
$i=3$				$n-3$
$\vdots$				$\vdots$
$i=n-1$				1

$$\underbrace{1} + 2 + \cdots + \underbrace{(n-1)} = \frac{n(n-1)}{2}$$