Prop 9.2.

If $\begin{cases} V \text{ vector space} \\ V \text{ generated by a } \underline{\text{finite}} \text{ subset } S \end{cases}$

Then there is a subset of S containing a basis, thus there is a finite basis for V.

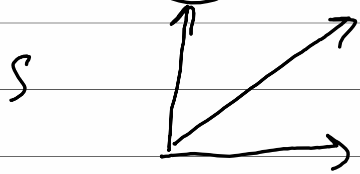
e.g. $\overbrace{\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}}$ in $\overset{(F^2)}{\mathbb{R}^2}$

These span $\mathbb{R}^2$ $\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}$ rank 2

<u>Not a basis</u> of $\mathbb{R}^2$

$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$ lin dep ↓.

$S = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$ spans $\mathbb{R}^2$



$\operatorname{span} S = \mathbb{R}^2$

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$$

$$\underline{\begin{pmatrix}1\\1\end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}}$$

Thus, $\underline{S} \subseteq \underline{\operatorname{span}\left\{ \underline{\begin{pmatrix}1\\0\end{pmatrix}}, \underline{\begin{pmatrix}0\\1\end{pmatrix}} \right\}}$

$$\operatorname{span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} = \mathbb{R}^2$$

as $\operatorname{span}(S) = \mathbb{R}^2$.

$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ spans $\mathbb{R}^2$

lin indep let $a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \vec{0}$

$$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \text{ so } \begin{matrix}a=0\\b=0\end{matrix}$$

Thus, this is a basis of $\mathbb{R}^2$

$\operatorname{Span}(S) = V$

Case 1. S lin indep

Thus, S is a basis for V

Case 2. S is lin dependent

Then there is $v \in S$ so that v is a linear combination of other elements of S

$$\underline{v} = a_1 \underline{v_1} + \dots + a_n v_n \quad , a_i \in F$$

$$v_i \in S \quad , \overline{v_i} \neq v$$

$$b_1 w_1 + \dots + b_k w_k = 0 \quad \text{for some } b_j \neq 0.$$

$$w_i \in S \quad , b_i \in F.$$

$$b_1 w_1 + \dots + \underbrace{b_j w_j} + \dots + b_k w_k = 0$$

$$-b_j w_j = b_1 w_1 + \dots + b_{j-1} w_{j-1} + b_{j+1} w_{j+1} + \dots + b_k w_k$$

$$w_j = \frac{-b_1}{b_j} w_1 + \dots + \frac{-b_{j-1}}{b_j} w_{j-1} - \frac{b_{j+1}}{b_j} w_{j+1} + \dots + \frac{-b_k}{b_j} w_k$$

Have $v \in S$ and $v_1, \dots, v_n \in S$

$a_1, \dots, a_n \in F$

so that $v_i \neq v$

and

$$v = a_1 v_1 + \dots + a_n v_n$$

$$\Updownarrow$$

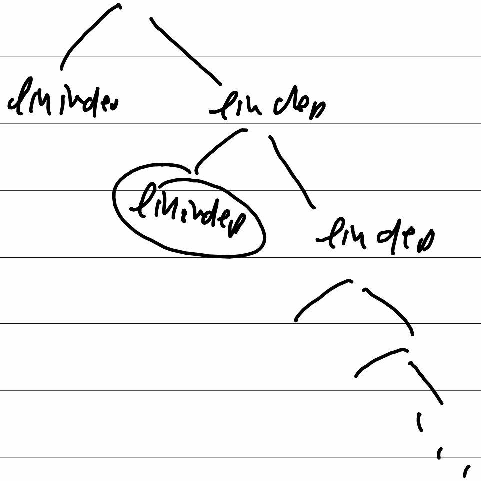
$$v \in \text{span}\{v_1, \dots, v_n\}$$

→ remove v from S

$v \in \text{span}(S \setminus \{v\})$ $v_i \neq v$.

$$S \subseteq \text{span}(\underbrace{S \setminus \{v\}}) = V$$

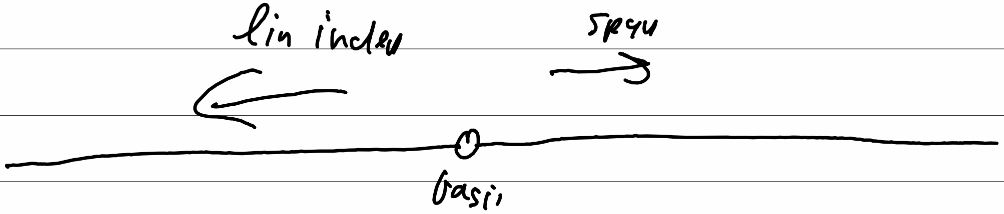
$$S \setminus \{v\} \subseteq S$$



ends as S is finit

Morale lin indep sets can be grown into bases

spanning sets can be shrunk to bases



2b. $\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \subseteq \underline{M_{2\times 2}(\mathbb{R})}$.

Is this lin indep?

Let $\underline{a} \cdot \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} + \underline{b} \cdot \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \overset{\swarrow ?}{0}$

$\left[\begin{array}{l} \text{If } \Rightarrow a = b = 0, \text{ then lin indep} \\ \text{If we find nontrivial } a, b, \text{ lin dep} \end{array}\right]$

$$\begin{pmatrix} a & -2a \\ -a & 4a \end{pmatrix} + \begin{pmatrix} -b & b \\ 2b & -4b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a-b & -2a+b \\ -a+2b & 4a-4b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a - b = 0 \rightarrow a = b$

$-2a + b = 0 \rightarrow -2a + a = 0, \quad -a = 0, \quad a = 0$

$-a + 2b = 0 \qquad a = 0 \Rightarrow b = 0$

$4a - 4b = 0$

Thus, $\underline{a = b = 0}$, so this is lin independent.

2d. $\{x^3 - x, \quad 2x^2+4, \quad -2x^3+3x^2+2x+6\}$

in $P(\mathbb{R})$

Let $a(x^3-x)+b(2x^2+4)+c(-2x^3+3x^2+2x+6)=0$

$(a, b, c \text{ in } \mathbb{R})$

$$\underline{(a-2c)x^3 + (2b+3c)x^2 + (-a+2c)x + (4b+6c)\cdot 1 = 0}$$

in $P(\mathbb{R})$

$\{1, x, x^3, x^2\}$ lin indep

$a-2c=0, \quad 2b+3c=0, \quad -a+2c=0, \quad 4b+6c=0$

$$\begin{array}{ccc|c} a & b & c & \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 4 & 6 & 0 \end{array}$$

$\to \mathbb{R}^4$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a - 2c = 0$$

$$2b + 3\underbrace{c}_{\text{free variable}} = 0$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2c \\ -\frac{3}{2}c \\ \underline{c} \end{pmatrix}$$

$c=2$

$$\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \text{ solution} \neq \vec{0}$$

$$4(x^3 - x) - 3(2x^3 + 4) + 2(-2x^3 + 3x^2 + 2x + 6) = 0$$

$$\cancel{(4x^3 - 4x^3)} + \cancel{(-6x^2 + 6x^2)} + \cancel{(-4x + 4x)} + \cancel{(-12 + 12)} = 0$$

Thus, this is a lin dep set.

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 4 & 6 \end{pmatrix}$$

$$\begin{array}{ccc} \nearrow & \uparrow & \uparrow \\ 1x^3-1\cdot x & 2x^2+4 & -2x^3+3x^2+2x+6 \end{array}$$

$$\underline{x^3-x} \longleftrightarrow \underline{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}$$

$$\underline{2x^2+4} \longleftrightarrow \underline{\begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}}$$

$$\underline{-2x^3+3x^2+2x+6} \longleftrightarrow \underline{\begin{pmatrix} -2 \\ 3 \\ 2 \\ 6 \end{pmatrix}}$$

"linear transformation"

$\Downarrow$

$$P_3(\mathbb{R}) \longleftrightarrow \mathbb{R}^4$$

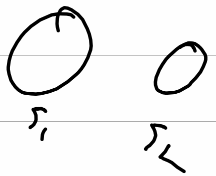
21. Let V vector space.

Let S_1, S_2 be subsets of V

Assume that S_1, S_2 lin indep

and $S_1 \cap S_2 = \emptyset$

disjoint



Prove that (($S_1 \cup S_2$ lin independent)) if and only if ($\text{span}(S_1) \cap \text{span}(S_2) = \{0\}$.)

Pf. ($\Rightarrow$) Let $S_1 \cup S_2$ lin indep.

wts. $\text{span}(S_1) \cap \text{span}(S_2) = \{0\}$.

Let $v \in \text{span}(S_1) \cap \text{span}(S_2)$

wts $v = 0$.

$v \in \text{span}(S_1)$ and $v \in \text{span}(S_2)$

$v = \sum_{i=1} a_i v_i$ and $v = \sum_{j=1}^{k} b_j w_j$

$a_i \in F$, $v_i \in S_1$ $\quad$ $b_j \in F$, $w_j \in S_2$

$$v = \underline{\sum_{i=1}^{n} a_i \overset{\in S_1}{v_i}} \quad \text{and} \quad v = \underline{\sum_{j=1}^{k} b_j \overset{\in S_2}{w_j}}$$

$$\text{Thus, } \underline{\sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{k} b_j w_j}$$

Know. $\boxed{S_1 \cap S_2 = \phi}$

and S_1, S_2 lin indep

$\boxed{\text{and } S_1 \cup S_2 \text{ lin indep}}$

$v_i \neq w_j$ for all i, j

S_1 S_2

$$\sum_{i=1}^{n} a_i \underbrace{v_i}_{S_1 \cup S_2} - \sum_{j=1}^{k} b_j \underbrace{w_j}_{S_1 \cup S_2} = 0$$

Thus, $a_i = 0$ and $b_j = 0$ all i, j
as $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2$ lin indep.

Thus $v = 0$, so $\text{Span}(S_1) \cap \text{Span}(S_2) = \{0\}$

($\Leftarrow$) Let $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2) = \{0\}$

wts $S_1 \cup S_2$ lin indep

Let $\underline{\sum a_i v_i + \sum b_j w_j = 0}$

$v_i \in S_1 \qquad\qquad w_j \in S_2$

$S_1 \cap S_2 = \emptyset$

$$\sum a_i v_i = -\sum b_j w_j$$

$$\rotatebox{90}{$\in$} \qquad\qquad\qquad \rotatebox{90}{$\in$}$$

$$\mathrm{span}(S_1) \quad \cap \quad \mathrm{span}(S_2) = \{0\}$$

$$\sum a_i \underbrace{v_i} = 0 = \sum b_j w_j$$

S_1 lin indep $\qquad\qquad$ S_2 lin indep

$\therefore a_i = 0 \qquad\qquad \therefore b_j = 0$

Thus, $S_1 \cup S_2$ lin indep. $\square$