OH

M 4–5

Th 3–4

Recordings my section in Canvas Discussions

Notes on my website

math.ucla.edu/~jas

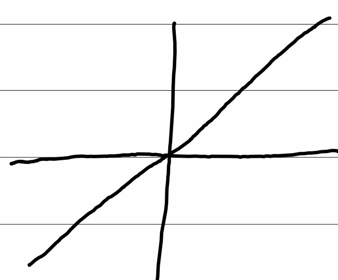
Error in notes $\overset{\in V}{v} = \overset{\in W_1}{w_1} + \dots + \overset{\in W_n}{w_n}$ uniquely

Def 6.3. $V = W_1 \oplus \dots \oplus W_n$

if $V = W_1 + \dots + W_n$

and $W_i \cap W_j = \{0\}$

False.



$W_i \cap \left(\sum_{i \neq j} W_j \right) = 0$

Vector ♡ Space $\left\{ \begin{pmatrix} a \\ \text{☺} \\ b \end{pmatrix} \middle| \begin{matrix} a \in \mathbb{R} \\ b \in \mathbb{R} \end{matrix} \right\}$

V ☺ $\longrightarrow$

Prove. $\mathbb{R} \times \{\text{☺}\} \times \mathbb{R}$ is a vector space $/\mathbb{R}$

w/ operations (a, b) ☺

$$\begin{pmatrix} a \\ \text{☺} \\ b \end{pmatrix} + \begin{pmatrix} a' \\ \text{☺} \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ \text{☺} \\ b+b' \end{pmatrix}$$

$$\lambda \begin{pmatrix} a \\ \text{☺} \\ b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \text{☺} \\ \lambda b \end{pmatrix}$$

Closure. $+ : V \times V \longrightarrow V \qquad v_1 + v_2 \in V$

$\cdot : \mathbb{R} \times V \longrightarrow V \qquad \lambda v \in V$

$\begin{pmatrix} a \\ \text{☺} \\ b \end{pmatrix} + \begin{pmatrix} a' \\ \text{☺} \\ b' \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} a+a' \\ \text{☺} \\ b+b' \end{pmatrix}$, an element of V

$\lambda \cdot \begin{pmatrix} a \\ \text{☺} \\ b \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \lambda a \\ \text{☺} \\ \lambda b \end{pmatrix}$, an element of V

Additive associativity,

want to show, for all v_1, v_2, v_3 in V that

$$v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$$

$$v_1 = \begin{pmatrix} a_1 \\ \vdots \\ b_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_2 \\ \vdots \\ b_2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} a_3 \\ \vdots \\ b_3 \end{pmatrix}$$

$$v_1 + (v_2 + v_3) = \begin{pmatrix} a_1 \\ \vdots \\ b_1 \end{pmatrix} + \left(\begin{pmatrix} a_2 \\ \vdots \\ b_2 \end{pmatrix} + \begin{pmatrix} a_3 \\ \vdots \\ b_3 \end{pmatrix} \right)$$

$$= \begin{pmatrix} a_1 \\ \vdots \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 + a_3 \\ \vdots \\ b_2 + b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + (a_2 + a_3) \\ \vdots \\ b_1 + (b_2 + b_3) \end{pmatrix}$$

$$(v_1+v_2)+v_3 = \left(\begin{pmatrix} a_1 \\ \vdots \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ \vdots \\ b_2 \end{pmatrix}\right) + \begin{pmatrix} a_3 \\ \vdots \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} (a_1+a_2) \\ \vdots \\ (b_1+b_2) \end{pmatrix} + \begin{pmatrix} a_3 \\ \vdots \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} (a_1+a_2)+a_3 \\ \vdots \\ (b_1+b_2)+b_3 \end{pmatrix}$$

Thus, $v_1+(v_2+v_3) = \begin{pmatrix} a_1+(a_2+a_3) \\ \vdots \\ b_1+(b_2+b_3) \end{pmatrix} = \begin{pmatrix} (a_1+a_2)+a_3 \\ \vdots \\ (b_1+b_2)+b_3 \end{pmatrix}$

$(v_1+v_2)+v_3 = \begin{pmatrix} (a_1+a_2)+a_3 \\ \vdots \\ (b_1+b_2)+b_3 \end{pmatrix}$ equal

So $v_1+(v_2+v_3) = (v_1+v_2)+v_3$, as addition over $\mathbb{R}$ is associative

So addition is associative.

Additive Commutativity,

For all v_1, v_2 in V, show

$$v_1 + v_2 = v_2 + v_1$$

Let $v_1 = \begin{pmatrix} a_1 \\ \vdots \\ b_1 \end{pmatrix}$, $v_2 = \begin{pmatrix} a_2 \\ \vdots \\ b_2 \end{pmatrix}$

$$\underline{v_1 + v_2} = \begin{pmatrix} a_1 \\ \vdots \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ \vdots \\ b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 \\ \vdots \\ b_1 + b_2 \end{pmatrix} \quad \text{def of } +$$

$$= \begin{pmatrix} a_2 + a_1 \\ \vdots \\ b_2 + b_1 \end{pmatrix} \quad \text{commutativity of addition} / \mathbb{R}$$

$$= \begin{pmatrix} a_2 \\ \vdots \\ b_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ b_1 \end{pmatrix} \quad \text{def of } +$$

$$= \underline{v_2 + v_1} \quad \text{def of } v_i$$

Thus, $v_1+v_2 = v_2+v_1$ so $+$ is commutative.

Additive identity.

want to show that there is some vector $\vec{0}$ in V so that for all v in V, $\vec{0}+v=v$.

real number

Indeed, $\vec{0} = \begin{pmatrix} 0 \\ \Downarrow \\ 0 \end{pmatrix}$

Let $v = \begin{pmatrix} a \\ \Downarrow \\ b \end{pmatrix}$

$$\vec{0}+v = \begin{pmatrix} 0 \\ \Downarrow \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ \Downarrow \\ b \end{pmatrix}$$

$$= \begin{pmatrix} 0+a \\ \Downarrow \\ 0+b \end{pmatrix} \quad \text{def of } +$$

$$= \begin{pmatrix} a \\ \Downarrow \\ b \end{pmatrix} \quad \text{because } a+0=a \text{ in } \mathbb{R}, \ b+0=b$$

$= v$

Thus, $\vec{0} + v = v$ for all v in V, so $\vec{0}$ is an additive identity.

Additive inverse,

For all v in V, there is w in V (dep on v) so that

$v + w = \vec{0}$

Indeed, let $v = \begin{pmatrix} a \\ \text{☺} \\ b \end{pmatrix}$ in V.

want to find w in V so that $v + w = \vec{0}$.

If $w = \begin{pmatrix} a' \\ \text{☺} \\ b' \end{pmatrix}$, this means $\begin{pmatrix} a \\ \text{☺} \\ b \end{pmatrix} + \begin{pmatrix} a' \\ \text{☺} \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ \text{☺} \\ 0 \end{pmatrix}$

$$\begin{pmatrix} a \\ \text{☺} \\ b \end{pmatrix} + \begin{pmatrix} a' \\ \text{☺} \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ \text{☺} \\ b+b' \end{pmatrix}$$

So we want this to equal $\begin{pmatrix} 0 \\ \text{☺} \\ 0 \end{pmatrix}$. Thus, we must take $a' = -a$, $b' = -b$.

That is, $w = \begin{pmatrix} -a \\ \text{☺} \\ -b \end{pmatrix}$ is an additive inverse too. Hence, all vectors have additive inverses in V.

Distributivity 1.

for λ, μ in $\mathbb{R}$ and v in V,

$$(\lambda+\mu)v = \lambda v + \mu v$$

Let $v = \begin{pmatrix} a \\ \shortparallel \\ b \end{pmatrix}$

$$(\lambda+\mu)v = (\lambda+\mu)\begin{pmatrix} a \\ \shortparallel \\ b \end{pmatrix}$$

$$= \begin{pmatrix} (\lambda+\mu)a \\ \shortparallel \\ (\lambda+\mu)b \end{pmatrix}$$ def of scalar mult on V

$$= \begin{pmatrix} \lambda a + \mu a \\ \shortparallel \\ \lambda b + \mu b \end{pmatrix}$$ distr. of real multiplication

$$= \begin{pmatrix} \lambda a \\ \shortparallel \\ \lambda b \end{pmatrix} + \begin{pmatrix} \mu a \\ \shortparallel \\ \mu b \end{pmatrix}$$ def of $+$

← def of scalar mult. on V

$$= \lambda\begin{pmatrix} a \\ \shortparallel \\ b \end{pmatrix} + \mu\begin{pmatrix} a \\ \shortparallel \\ b \end{pmatrix} = \lambda v + \mu v$$

Distr. 2

For λ in $\mathbb{R}$ and v_1, v_2 in V

$$\lambda(v_1+v_2) = \lambda v_1 + \lambda v_2$$

Let $v_i = \begin{pmatrix} a_i \\ \vdots \\ b_i \end{pmatrix}$ $i = 1, 2$

$$\lambda(v_1+v_2) = \lambda\left(\begin{pmatrix} a_1 \\ \vdots \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ \vdots \\ b_2 \end{pmatrix}\right)$$

$$= \lambda\begin{pmatrix} a_1+a_2 \\ \vdots \\ b_1+b_2 \end{pmatrix} \quad \text{def of } +$$

$$= \begin{pmatrix} \lambda(a_1+a_2) \\ \vdots \\ \lambda(b_1+b_2) \end{pmatrix} \quad \text{def of } \cdot$$

$$= \begin{pmatrix} \lambda a_1+\lambda a_2 \\ \vdots \\ \lambda b_1+\lambda b_2 \end{pmatrix} \quad \text{distr } /\mathbb{R}$$

$$= \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda b_1 \end{pmatrix} + \begin{pmatrix} \lambda a_2 \\ \vdots \\ \lambda b_2 \end{pmatrix} = \lambda v_1 + \lambda v_2$$

Mult. associativity,

$$\lambda(\mu v) = (\lambda\mu)v$$

$$\lambda\left(\mu\begin{pmatrix} a \\ \vdots \\ b \end{pmatrix}\right) = \lambda\begin{pmatrix} \mu a \\ \vdots \\ \mu b \end{pmatrix} = \begin{pmatrix} \lambda\mu a \\ \vdots \\ \lambda\mu b \end{pmatrix} = (\lambda\mu)v$$

$1 \cdot v = v$

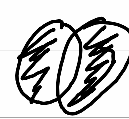
$$1 \cdot \begin{pmatrix} a \\ \vdots \\ b \end{pmatrix} = \begin{pmatrix} 1\cdot a \\ \vdots \\ 1\cdot b \end{pmatrix} = \begin{pmatrix} a \\ \vdots \\ b \end{pmatrix}$$

X set

Challenge. $P(X) = \{A \mid A \subseteq X\}$

Show that $P(X)$ is a vector space / $\mathbb{F}_2$

w/ operations

addition $\left[\, A+B = (A \cup B) - (A \cap B)\right.$ 

Scalar $\begin{cases} 0 \cdot A = \emptyset \\ 1 \cdot A = A \end{cases}$

Linear independence

Let V be a vector space

$S \subseteq V$ is lin independent if

$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = 0$ for $a_i \in F$ and $v_i \in S$

implies $a_1 = \cdots = a_n = 0$

$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ lin indep $/\mathbb{R}$ in $\mathbb{R}^3$ w/ componentwise

(row reduction)

$V =$ set of smooth function $\mathbb{R} \longrightarrow \mathbb{R}$

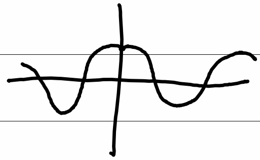
vs $/\mathbb{R}$ w/ ptwise addition and scalar problem

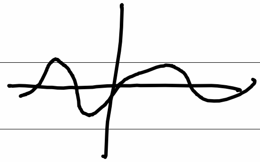
$(f+g)(x) = f(x) + g(x)$

$(\lambda f)(x) = \lambda f(x)$

constant function $\to 0$ is the addition identity

$\cos(x) \in V$, $\sin(x) \in V$





Claim. $\{\cos(x), \sin(x)\}$ linearly independent subset of V

Pf. Let $\underbrace{\lambda\cos(x) + \mu\sin(x)}_{\text{This is } 0 \text{ for all } x \text{ in } \mathbb{R}} = 0$ ← constant fn at 0

as functions

Goal is to show $\lambda = 0$ and $\mu = 0$

Evaluate at $x = 0$

$$\lambda\cos(0) + \mu\sin(0) = 0$$
$$\lambda \cdot 1 + \mu \cdot 0 = \lambda$$

Eval. @ $x = \frac{\pi}{2}$

$$\lambda\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{0} + \mu\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1} = 0$$
$$= \mu$$

Thus, $\lambda = 0$ and $\mu = 0$ so $\{\cos(x), \sin(x)\}$ is lin indep in V.

$V =$ smth fns $\mathbb{R} \longrightarrow \mathbb{R}$

$W = \{ f \in V \mid f(x) = f(-x) \}$

$\left(\text{Span}\{\cos(x), \sin(x)\} \right) \cap W \subseteq V$ subspace

Let $f \in$.

$f \in \text{Span}\{\cos(x), \sin(x)\}$

so $f(x) = \lambda \cos(x) + \mu \sin(x)$

also, $f \in W$ so $f(x) = f(-x)$

$$\lambda \cos(x) + \mu \sin(x) = \lambda \cos(-x) + \mu \sin(-x)$$

$$= \lambda \cos(x) - \mu \sin(x)$$

$\mu \sin(x) = -\mu \sin(x)$

$2\mu \sin(x) = 0$

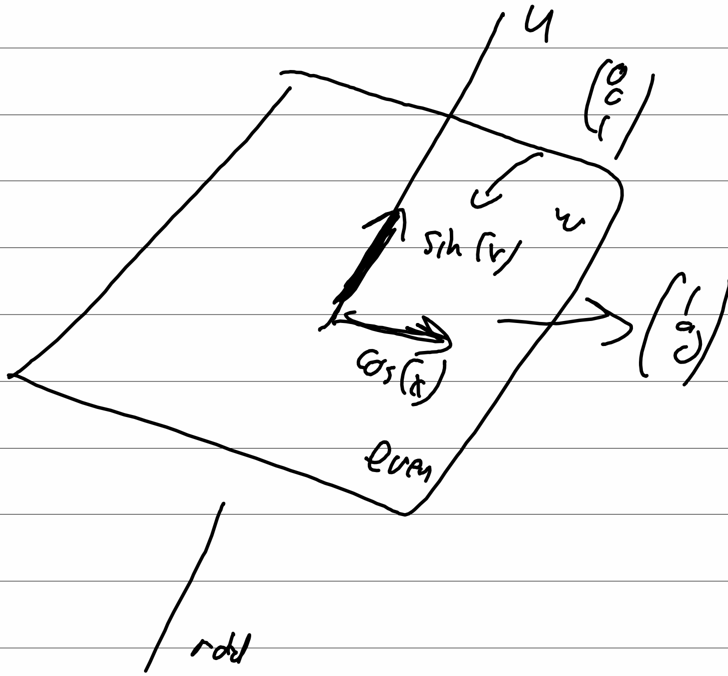
$2\mu = 0, \quad \mu = 0.$

$f(x) = \lambda \cos(x).$

$= \text{Span}\{\cos(x)\}$

$U =$ odd fns

$$\text{Span}\{\overset{\nearrow \text{even}}{\cos(x)}, \overset{\nearrow \text{odd}}{\sin(x)}\} \cap U = \text{Span}\{\sin(x)\}$$



$V = \mathbb{R}^3$, $W =$ xy-plane, $U =$ z axis

$$\left(\text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\right) \cap W = \text{Span}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\left(\text{Span}\left\{\underset{W}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \underset{U}{\begin{pmatrix}0\\0\\1\end{pmatrix}}\right\}\right) \cap U = \text{Span}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

V is the set of all diff. real valued fns on $\underbrace{\text{the real line}}_{\mathbb{R}}$

$$V = \{ f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ differentiable} \}$$

Show V is a VS w/ addition and scalar mult. defined as

$$(f, g) \longmapsto f+g \qquad (f+g)(x) = \frac{f(x)^{g(x)-e}}{17.3}$$
$$(\lambda, f) \longmapsto \lambda f$$

f, g in V — sum of fns — of real #'s

Def. $(f+g)(x) = f(x) + g(x) \qquad x \longmapsto f(x)+g(x)$

$$(f+g)(s) = f(s) + g(s)$$

scalar — mult of real #'s

$$(\lambda f)(x) = \lambda \cdot f(x))$$

$f+g$ in $\boxed{F(\mathbb{R}, \mathbb{R})}$

Closure. f, g in V

$\Downarrow$

$f+g$ in V

Issue: $f+g$ is a function $\mathbb{R} \longrightarrow \mathbb{R}$

$f+g$ in V?

V = Set of all diff. real val fn on $\mathbb{R}$

$= \{ f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ differentiable} \}$

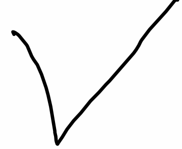
$W = \{ \cos(x) \}$

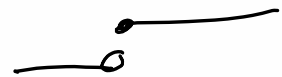
$\cos(x) + \cos(x) \notin W$

$f, g \in V$

$\Downarrow$

wts $f+g \in V$





$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$

$\cancel{f(x)} = 0$

$f+g : \mathbb{R} \longrightarrow \mathbb{R}$

is $f+g$ differentiable?

f, g are differentiable, as by hypothesis f, g in V and V is the set of differentiable fns $\mathbb{R} \longrightarrow \mathbb{R}$

$f, g \in V \implies f+g \in V$

to show $f+g$ in V, wts $f+g$ differentiable

$f+g$ differentiable as f, g differentiable

Closure under addition.

Let f, g in V

$(f+g)(x) = f(x) + g(x)$.

"fn"

f+g is thus a function $\mathbb{R} \longrightarrow \mathbb{R}$

Furthermore, f, g diff. as they are assumed to be in V

Thus, as the sum of diff fns is diff (Cs Calculus)

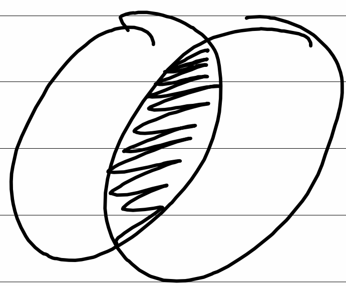
f+g diff

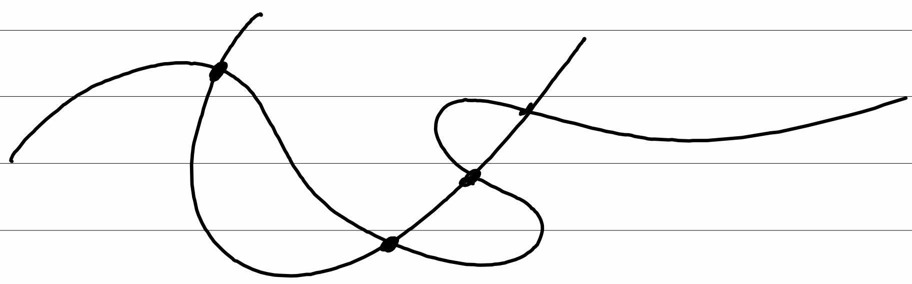
Hence, by def of V, f+g in V

$W_4 = \{ a_1 - 4a_2 - a_3 = 0 \}$

$W_3 = \{ 2a_1 - 7a_2 + a_3 = 0 \}$

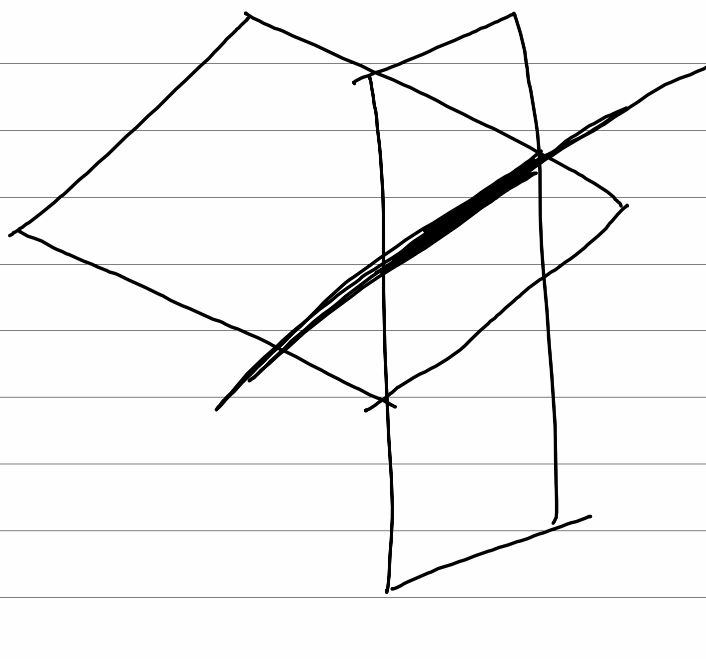
$W_3 \cap W_4 = \{ x \in \mathbb{R}^3 \mid x \in W_3 \ \underbrace{\text{and}} \ x \in W_4 \}$





$$W_3 \cap W_4 = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} (a_1, a_2, a_3) \in W_3 \\ \text{and} \\ (a_1, a_2, a_3) \in W_4 \end{array} \right\}$$

$$= \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 2a_1 - 7a_2 + a_3 = 0 \\ \text{and} \\ a_1 - 4a_2 - a_3 = 0 \end{array} \right\}$$

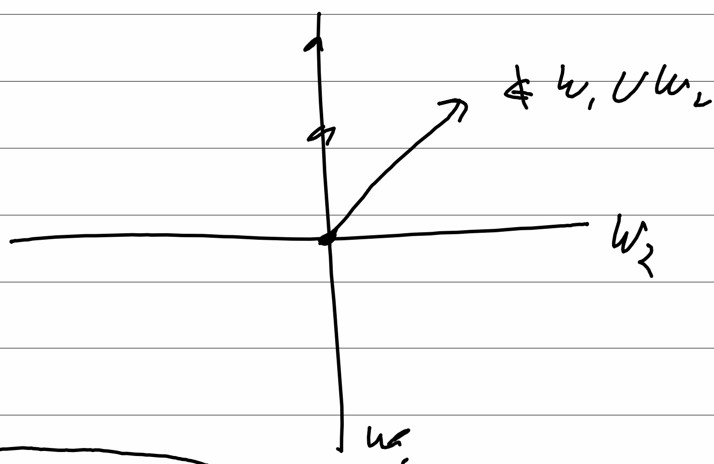


- $0 \in W$
- W closed under $+$
- W closed under scalar mult

Prove. $W_1 \cup W_2$ subspace

$$\Updownarrow$$

$W_1 \subseteq W_2$ or $W_2 \subseteq W_1$



- 0 in $W_1 \cup W_2$ T
- scalar mult T
- addition F

$W_1 \cup W_2$ subspace $\Rightarrow$ $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

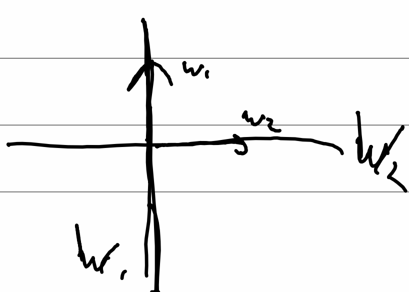
By contradiction, suppose

not ($W_1 \subseteq W_2$)
and
not ($W_2 \subseteq W_1$)

"$W_1 \subseteq W_2$" $\Leftrightarrow$ "for all $w \in W_1$, $w \in W_2$"

not $(W_1 \subseteq W_2)$ $\Leftrightarrow$ there exists $w_1 \in W_1$ s.t. $w_1 \notin W_2$

not$(W_2 \subseteq W_1)$ $\Leftrightarrow$ " " $w_2 \in W_2$ s.t. $w_2 \notin W_1$



Suppose not — — —

Hypotheses

w_1 in $W_1 - W_2$

w_2 in $W_2 - W_1$

W_1, W_2 subspaces

$W_1 \cup W_2$ subspace

→ 0

+

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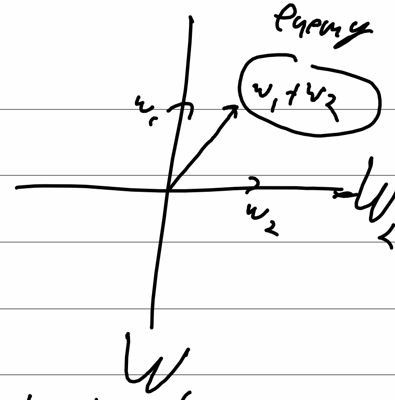


$w_i \in W_1 \cup W_2$

$w_1 + w_2 \notin W_1 \cup W_2$

∴ not a subspace

goal: contradict that $W_1 \cup W_2$ subspace

(addition)

$$w_1, w_2 \in W_1 \cup W_2$$

want $w_1 + w_2 \notin W_1 \cup W_2$

Suppose not. That is, what if $w_1 + w_2 \in W_1 \cup W_2$?

$$w_1 + w_2 \in W_1 \cup W_2$$

$$w_1 + w_2 \in W_1 \quad \text{or} \quad w_1 + w_2 \in W_2$$

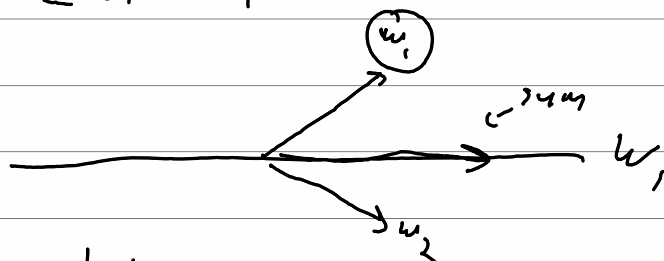
$w_1 \in W_1 - W_2$, $w_2 \in W_2 - W_1$

W_1, W_2 subspaces

Let's consider the case

$$w_1 + w_2 \in W_1$$

$$\begin{cases} w_2 \notin W_1 \\ w_1 \in W_1 \end{cases}$$



$$-w_1 \in W_1$$

$$\underbrace{\underbrace{(w_1+w_2)}_{\text{in } W_1} - \underbrace{w_1}_{W_1}}_{W_1} = w_2 \to \text{not in } W_1$$

Contradiction

Thus, $w_1 + w_2 \notin W_1$
〃 〃 $\notin W_2$

$w_1 + w_2 \notin W_1 \cup W_2$