13.1 Vectors

1. Compute the components of the vector $\overrightarrow{PQ}$ for

a) $P = (1,2)$, $Q = (3,4)$

b) $P = (3,4)$, $Q = (1,2)$

c) $P = (0,5,-10)$, $Q = (-5,7,-9)$

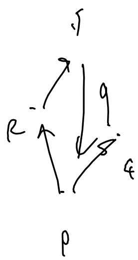
2. Draw the vector $\overrightarrow{PQ}$ on the xy-plane from parts (a) and (b) from 1.

3. a) Let $P = (1, 1)$, $Q = (2, 3)$, $R = (-1, 2)$, $S = (4, 2)$

Draw $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{QS}$, $\overrightarrow{RS}$ on the same plot

b) Compute the components of

$$\overrightarrow{PQ} + \overrightarrow{PR}$$



and draw it on the same plot with its tail at P

c) Compute the components of

$\overrightarrow{SQ}$, $\overrightarrow{QP}$, and $\overrightarrow{SP}$

How do they appear on your drawing?

4. Compute the components of the following vectors and draw them on the same plot with tail at the origin.

a) $\vec{v} = \langle 1,1 \rangle + \langle 2,2 \rangle$

b) $\vec{w} = 3\langle -1,-2 \rangle$

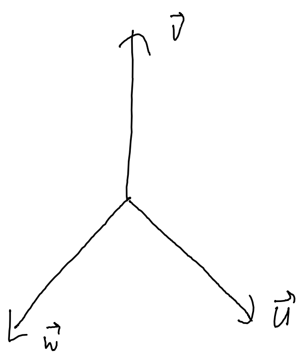
c) $\vec{v} + \vec{w}$

d) $3\vec{v} - 2\vec{w}$

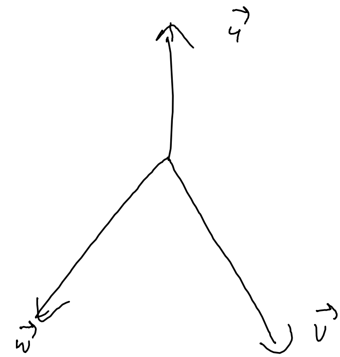
13.2 | 3D Space

1. For which of the following is $\{\vec{u}, \vec{v}, \vec{w}\}$ a right handed system?

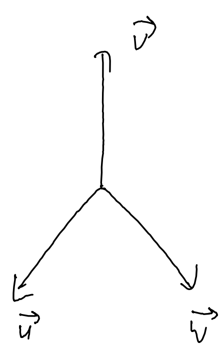
a)



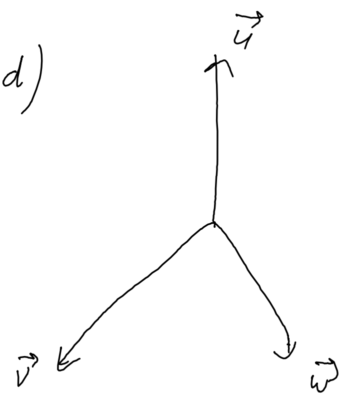
b)



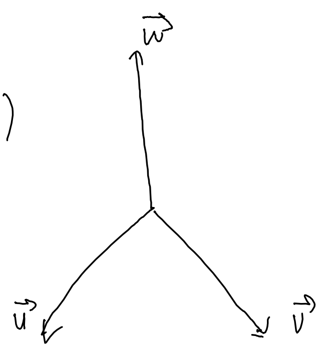
c)



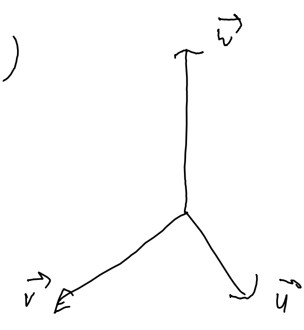
d)



e)



f)



2. If $\{\vec{u}, \vec{v}, \vec{w}\}$ is a right handed system, which of the following are also right handed?

a) $\{\vec{v}, \vec{u}, \vec{w}\}$

b) $\{\vec{v}, \vec{w}, \vec{u}\}$

c) $\{\vec{u}, \vec{w}, \vec{v}\}$

3. Let $\vec{v} = \langle 1, 2, 3 \rangle$

$\vec{w} = \langle -1, 0, 5 \rangle$

a) Compute $\|\vec{v}\|$

b) Compute $\|-3\vec{v}\|$

c) Are $\vec{v}$ and $\vec{w}$ parallel?

13.3 | Dot products

1. Let $\vec{v} = \langle 1, 2, 3 \rangle$, $\vec{w} = \langle -1, -3, 2 \rangle$,

$\vec{u} = \langle 5, 5, 5 \rangle$. Compute

a) $\vec{v} \cdot \vec{w}$

b) $\vec{v} \cdot \vec{u}$

c) $\vec{w} \cdot \vec{u}$

d) $\vec{v} \cdot (\vec{w} + \vec{u})$

e) $\vec{v} \cdot \vec{v}$

2. Given $\vec{v}$, $\vec{w}$, $\vec{u}$ from problem 1,

a) Is the angle between $\vec{v}$ and $\vec{w}$ obtuse, acute, or right? Compute the angle.

b) Same question for $\vec{v}$ and $\vec{u}$.

c) Same question for $\vec{u}$ and $\vec{w}$.

3. Let $\vec{v} = \langle 3, -7, 100 \rangle$.

Compute

a) $\vec{v} \cdot \langle 1, 0, 0 \rangle$

b) $\vec{v} \cdot \langle 0, 1, 0 \rangle$

c) $\vec{v} \cdot \langle 0, 0, 1 \rangle$

What do you notice?

4. Compute the projection of $\langle 3, -7, 100 \rangle$

onto

a) $\langle 1, 0, 0 \rangle$

b) $\langle 0, 1, 0 \rangle$

c) $\langle 0, 0, 1 \rangle$

5. a) Compute the projection of $\vec{v} = \langle 1, 2 \rangle$ along $\vec{u} = \langle -1, -5 \rangle$.

b) Draw $\vec{v}$, $\vec{u}$, and your answer from (a) on the same plot.

c) Compute $\vec{v}_{\perp \vec{u}}$

d) Compute the decomposition of $\vec{v}$ with respect to $\vec{u}$.

13.4 | Cross products

1. Compute the following determinants.

a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

b) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

c) $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$

d) $\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$

2. Compute the following cross products.

a) $\langle 1, 2, 1 \rangle \times \langle 2, 4, 3 \rangle$

b) $\langle 2, 4, 3 \rangle \times \langle 1, 2, 1 \rangle$

c) $\langle 1, 0, 0 \rangle \times \langle 2, 4, 3 \rangle$

d) $\langle 0, 1, 0 \rangle \times \langle 2, 4, 3 \rangle$

e) $\langle 0, 0, 1 \rangle \times \langle 2, 4, 3 \rangle$

3. Verify that $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$ for all parts of problem 2.

4. Verify that $\{\vec{v}, \vec{w}, \vec{v} \times \vec{w}\}$ is a right handed system for all parts of problem 2.

13.5 | Planes

1. Find a normal vector for the planes given by the following equations

a) $3x+4y-6z=0$

b) $3x+4y-6z=10\pi$

c) $x-y=z-10$

d) $x=z-y+3x-10^{10^{10}}$

2. Find a point on the planes described in problem 1.

3. Write an equation for the planes with the following normal vectors and points.

a) $\vec{n} = \langle 1, 2, 3 \rangle$, $p = (0, 0, 0)$

b) $\vec{n} = \langle 1, 2, 3 \rangle$, $p = (-10, 5, 7)$

c) $\vec{n} = \langle -1, 0, 15 \rangle$, $p = (-10, 5, 7)$

3. Write an equation for the planes containing the following three points

a) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

b) $(1,0,-1)$, $(2,2,1)$, $(4,1,2)$

12.1 | Parametric Equations

1. What shape do the following parametric curves describe?

a) $\vec{r}(t) = (\cos(t), \sin(t))$

b) $\vec{r}(t) = (\sin(t), \cos(t))$

c) $\vec{r}(t) = (1+3\sin(t), -2+3\cos(t))$

d) $\vec{r}(t) = (2\cos(t), 3\sin(t))$

e) $\vec{r}(t) = (1+t, 3+4t)$

f) $\vec{r}(t) = (1+t, t^2)$

2. Sketch the plots from each part of problem 1.

3. Find a parametrization for the lines passing through the point p with direction vector $\vec{v}$

a) $p = (0,0,0)$, $\vec{v} = \langle 1,2,3 \rangle$

b) $p = (-1,5,7)$, $\vec{v} = \langle 0,-10,17 \rangle$

4. Find a parametrization for the lines passing through the following two points.

a) $(0,0,0)$, $(1,1,1)$

b) $(1,2,3)$, $(4,5,6)$

c) $(-1,0,-10)$, $(1,17,-3)$

5. Find a paramtrization for the circles with radius r and center P.

a) $r = 1$, $P = (0,0)$

b) $r = 17$, $P = (1,2)$

c) $r = \sqrt{2}$, $P = (10,10)$

Are your parametrizations clockwise or counterclockwise?

How would you modify your parametrization to change that?

14.1 | Vector Valued Functions

1. Describe the projections of

$$\vec{r}(t) = \langle \cos(t), t, \sin(t) \rangle$$

on the xy, xz, and yz planes.

2. Compute the domain of

$$\vec{r}(t) = \left\langle \sqrt{t},\ \ln(1-t),\ \frac{1}{t^3} \right\rangle$$

14.2 | Calculus of vector valued functions

1. Compute the following limits

a) $\lim_{t \to 2} \langle t, t^2, t^3 \rangle$

b) $\lim_{t \to 0} \langle t, t^2, \frac{1}{t} \rangle$

c) $\lim_{t \to \infty} \langle \frac{1}{t}, \frac{2}{t}, \frac{3}{t} \rangle$

2. Compute $\vec{r}\,'(t)$ for the following functions

a) $\vec{r}(t) = \langle t, \cos(t), e^{-t} \rangle$

b) $\vec{r}(t) = \langle \ln(t), \sin(t) \rangle$

3. Compute the following derivatives using differentiation rules.

a) $\frac{d}{dt}\left(\frac{1}{\sqrt{t^2+1}}\langle 3t^3, e^{-t^2}\rangle\right)$

b) $\frac{d}{dt}\langle \cos(t^2), \sin(t^2)\rangle$

c) $\frac{d}{dt}\left(\langle \cos(t), \sin(t)\rangle \cdot \langle t, t^2\rangle\right)$

4. Parametrize the tangent vector to $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $\langle 0,0,0 \rangle$ and at $\langle 1,1,1 \rangle$.

5. Compute the following integrals.

a) $\int_0^1 \langle t, t^2, t^3 \rangle \, dt$

b) $\int_0^{2\pi} \langle \cos(t), \sin(t), e^t \rangle \, dt$

6. Compute the following via the fundamental theorems of calculus

a) $\int_0^1 \langle 1, 2t, 3t^2 \rangle \, dt$

b) $\frac{d}{dt} \int_0^t \langle \cos(u), e^{\sin(u)} \rangle \, du$

14.3 | Arc length and speed

1. Compute the arc length parametrization of $\langle \cos(t), \sin(t), t \rangle$ from the point $\langle 1, 0, 0 \rangle$.

2. Is $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ an arc length parametrization?

||

3. Which of the following is an arc length parametrization of a circle of radius 2 centered at $(1,1)$?

a) $\langle 1+2\sin(t), 1+2\cos(t)\rangle$

b) $\langle 2\sin(2t), 2\cos(2t)\rangle$

c) $\langle 1+2\sin(2t), 2\cos(2t)\rangle$

d) $\langle 1+2\cos\left(\frac{t}{2}\right), 1+2\sin\left(\frac{t}{2}\right)\rangle$

e) $\langle 1+2\cos\left(\frac{t}{\sqrt{2}}\right), 1+2\sin\left(\frac{t}{\sqrt{2}}\right)\rangle$

14.4 | Curvature

1. Compute $\kappa(t)$ for the following curves $\vec{r}(t)$.

a) $\vec{r}(t) = \langle 1+t, 2+3t, 17+5t \rangle$

b) $\vec{r}(t) = \langle t, t, t \rangle$

c) $\vec{r}(t) = \langle 1+\cos(t), 1+5\sin(t) \rangle$

d) $\vec{r}(t) = \langle \sqrt{2}\cos(t), \sqrt{2}\sin(t) \rangle$

e) $\vec{r}(t) = \langle \frac{1}{2}\sin(t), \frac{1}{2}\cos(t) \rangle$

f) $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

2. Compute the unit tangent and normal vectors for the following curves.

a) $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$

b) $\vec{r}(t) = \langle \sin(t), \cos(t) \rangle$

c) $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

d) $\vec{r}(t) = \langle -t, t^2, -t^3 \rangle$

3. Compute the oscalating plane and circle to $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $\langle 0, 0, 0 \rangle$ and $\langle 1, 1, 1 \rangle$.

14.5 | Motion in 3-Space

1. a) Compute the acceleration vector of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t=0$ and $t=1$.

b) Decompose the acceleration vectors you found into their tangential and normal components.

2. Consider $\vec{r}(t) = \langle \cos(t^3), \sin(t^3) \rangle$

a) What shape does $\vec{r}(t)$ parametrize?

b) Draw the unit tangent and normal vectors at $t = -1, 0, 1$

c) For what values of t is the speed decreasing? Increasing? Momentarily constant?

d) Draw the acceleration vectors at $t = -1, 0, 1$ on your drawing from (b).

e) How does your drawing relate to your answer in (c)?

13.6 | Quadric Surfaces

1. Draw the x, y, and z traces of the following quadric surfaces.

a) $z^2 = x^2 + y^2$

b) $x^2 = 2y^2 - z^2$

c) $x^2 + 2y^2 + 3z^2 = 4$

d) $x = y^2 + z^2$

e) $y = z^2 - 2x^2$

2. Draw the quadric surfaces from the previous part and name them.

15.1 | Functions of several variables

1. Compute the domains of the following functions.

a) $f(x,y) = \sqrt{x} + \sqrt{y}$

b) $f(x,y) = \sqrt{1-x^2-y^2}$

c) $f(x,y) = \dfrac{\ln(x)}{\sqrt{y}}$

d) $f(x,y,z) = \ln(xy) + z$

2. Sketch vertical traces of the following functions.

a) $f(x,y) = y\sin(x)$

b) $f(x,y) = x^2+y^2$

c) $f(x,y) = x-y^2$

3. Plot level curves of the same 3 functions from problem 2.

4. Match the functions to their level curves.

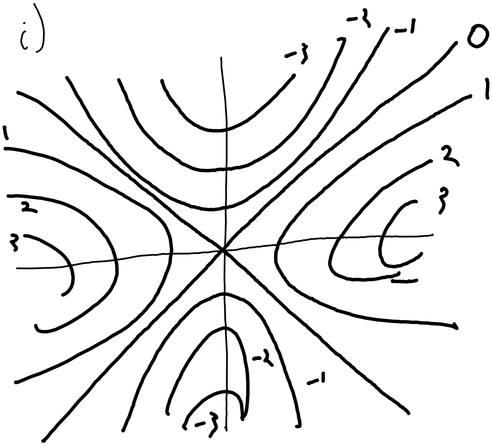
a) $f(x,y) = x+y$

b) $f(x,y) = x-y$

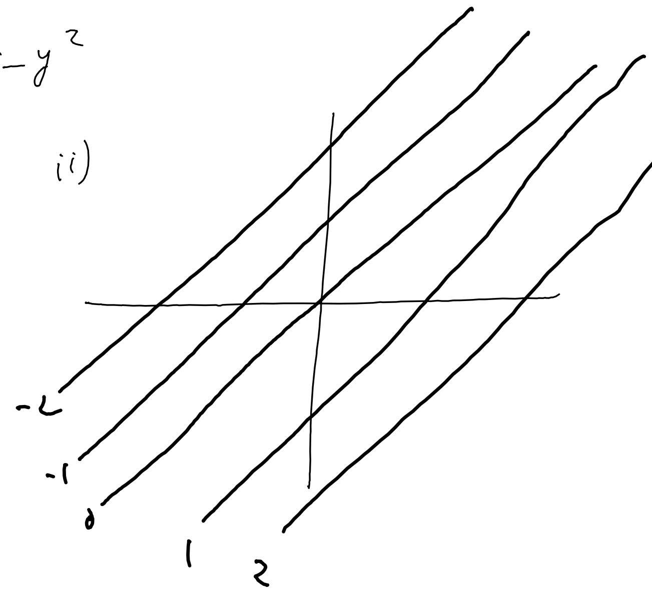
c) $f(x,y) = x^2+y^2$

d) $f(x,y) = x^2-y^2$

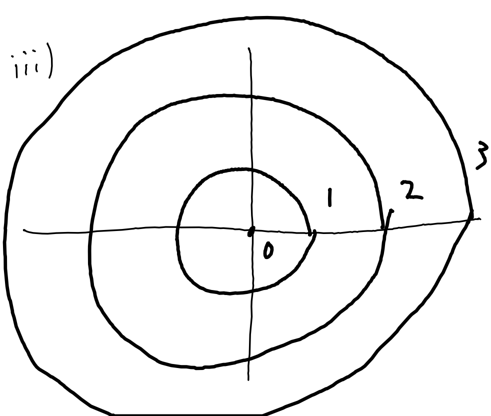
i)



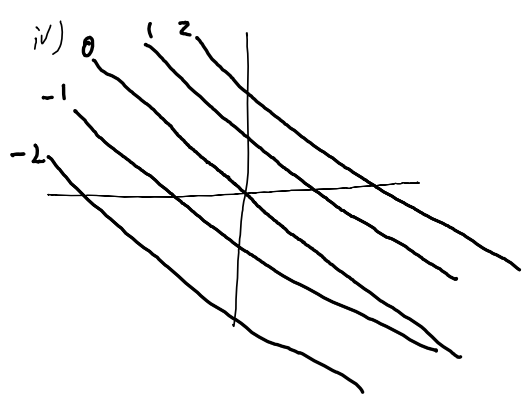
ii)



iii)



iv)



15.2 Limits and Continuity in Several Variables

1. Compute the following limits

a) $\lim_{(x,y)\to(1,2)} x^2 + y - xy$

b) $\lim_{(x,y)\to(1,0)} x \frac{\sin(y)}{y}$

c) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

2. Compute the following limits or show they don't exist:

a) $$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

b) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

c) $$\lim_{(x,y)\to(1,1)} \frac{2x^2-xy-y^2}{x^2-y^2}$$

d) (Tough!) $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

15.3 | Partial derivatives

1. Compute the following partial derivatives

a) $\frac{\partial}{\partial x}(x^2+y^2)$

b) $\frac{\partial}{\partial y}(xy)$

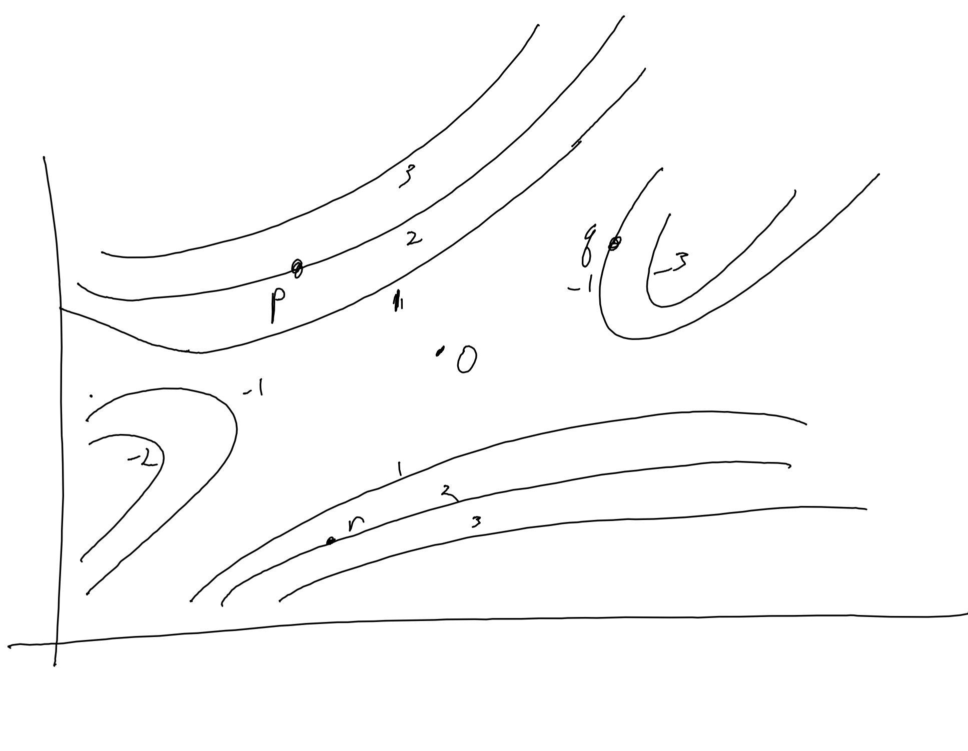
c) $\frac{\partial}{\partial z}\left(xz+y^2\ln(x\sin(e^{-\cos(xy)}))\right)$

2. Compute the following higher order partial derivatives.

a) f_{zzwx} for $f(x,y,z,w) = x^3 w^2 z^2 + \sin\left(\frac{xy}{z^2}\right)$.

b) g_{uuv} for $g(u,v) = \cos(u+v^2)$

3. Determine if the partial derivatives at the specified points are positive or negative given the contour plot.



15.4 | Tangent planes and linear approximation

1. Find an equation for the tangent planes for the following functions at the specified points.

a) $f(x,y) = x^2 + y^2 - xy$ at $(1, 1, f(1,1))$

b) $f(x,y) = \cos(x)\sin(y)$ at $(0, 0, f(0,0))$

c) $f(x,y) = e^{xy}$ at $(1, 1, f(1,1))$

2. Compute the linear approximations of the previous problem's functions near the specified points.

3. Use linearization to approximate $(2.92)^3\sqrt{4.08}$

15.5 | The gradient and directional derivatives

1. Compute ∇f for the following functions.

a) $f(x,y) = x^2 - y^2$

b) $f(x,y) = (x+y)^{100}$

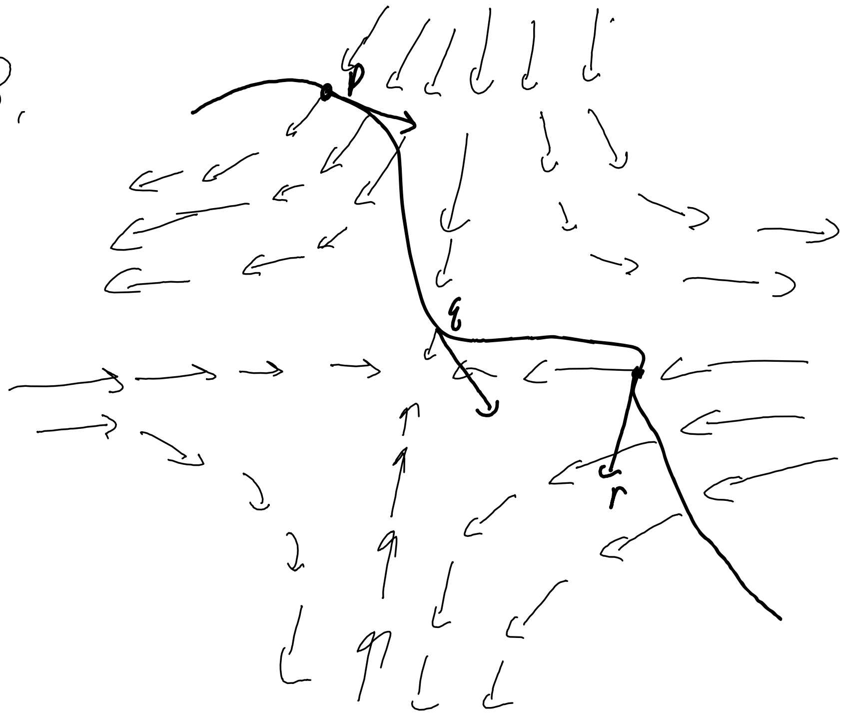
c) $f(x,y) = \cos(xy)$

2. Plot gradient fields for the following functions.

a) $f(x,y) = x^2 - y^3$

b) $f(x,y) = x + y$

3.



The plot is a vector field of ∇f with a curve $\vec{r}(t)$. Is $f(\vec{r}(t))$ increasing, decreasing, or instantaneously constant at the points p, q, and r?

The drawn arrows are the velocity vectors of $\vec{r}$.

4. Let $f(x,y) = x^3+y^3$

Compute the directional derivatives at the following points and directions

a) $p = (2,3)$, $\vec{u} = \langle 1, 0 \rangle$

b) $p = (2,3)$, $\vec{u} = \langle 0, 1 \rangle$

c) $p = (2,3)$, $\vec{u} = \langle -1, 0 \rangle$

d) $p = (2,3)$, $\vec{u} = \langle 0, -1 \rangle$

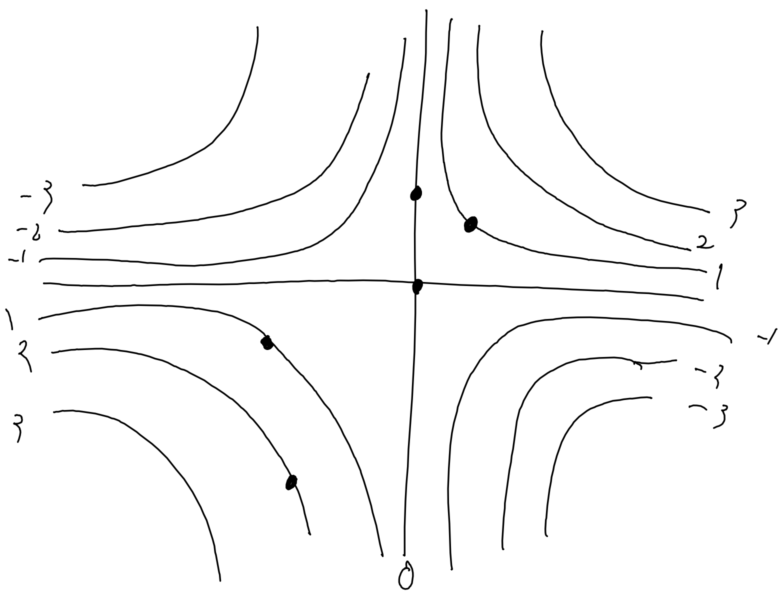
e) $p = (1,-1)$, $\vec{u} = \langle 1, 1 \rangle$

In each case, is f increasing or decreasing at p in the $\vec{u}$ direction?

5. Let f be a function of several variables differentiable at P. In which direction is f increasing fastest? decreasing fastest?

6. Here is a contour plot for

$f(x, y) = xy$.



Sketch gradients at the specified points

7. Find an equation for the tangent plane to the surface $x^2+y^2=z^2$ at $p=(3,4,5)$

15.6 | Multivariable chain rules

1. Compute the following composite functions

a) $f(x,y) = x^2 - y^2$, $x = r\cos(\theta)$, $y = r\sin(\theta)$

b) $f(x,y) = \frac{x}{y}$, $x = s+u$, $y = st$, $z = e^{stu}$

2. Draw diagrams representing the relationships between variables in the examples from (1).

3. Compute the partial derivatives of the composites from (1).

4. Let $f(x,y) = \dfrac{G m_1 m_2}{x^3+y^2}$

where G, m_1, and m_2 are constants.

a) Draw a contour plot of f

b) Draw a gradient plot of f on top of your contour plot.

c) Compute $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$

$f(x,y)$ is the force exterted on a particle of mass m_1 at (x,y) via gravity from a particle of mass m_2 at the origin.

15.7 Optimization

1. Compute the critical points of the following functions.

a) $f(x,y) = (x^2+y^2)e^{-x}$

b) $f(x,y) = x^3+y^3-12xy$

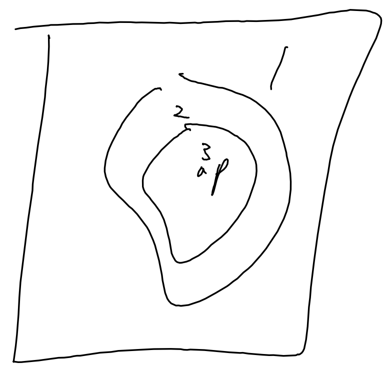
c) $f(x,y) = 3xy^2-x^3$

2. Apply the 2^{nd} derivative test to analyze the critical points of the previous problem's functions.

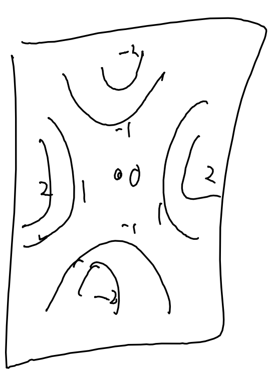
3. Are the marked points on the following contour maps local minima, maxima, saddle points, or none of the above?

a)

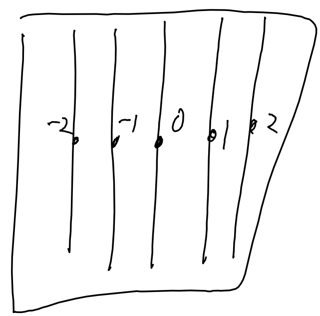
b)



c)



d)



4. Find the extremal values of $f(x,y) = x^2 + y^2$ on the box where $-1 \le x \le 1$ and $-1 \le y \le 1$.

15.8 | Lagrange multipliers

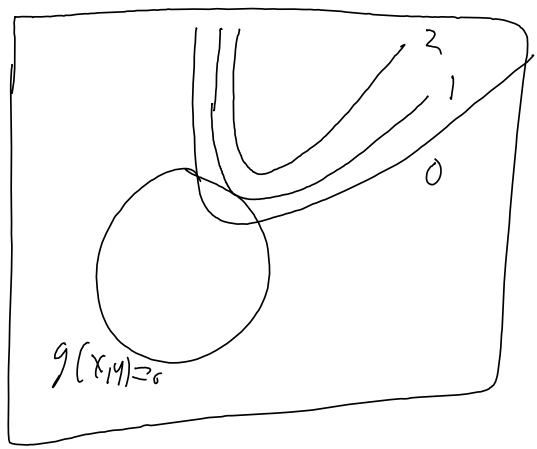
1. Find the extreme values of the following functions with the following constraints

a) $f(x,y) = 2x+5y$ on $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

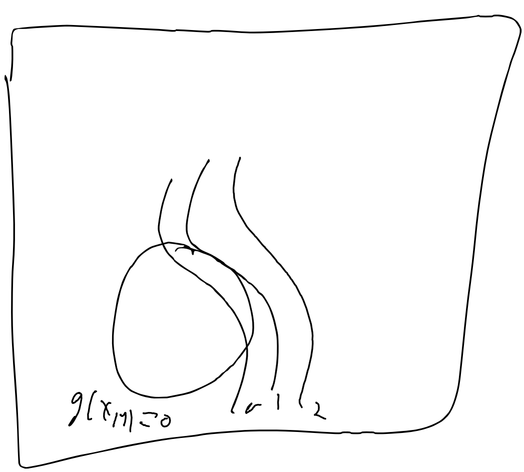
b) $f(x,y) = x^2+y^2$, $g(x,y) = x-y = 1$

2. Is the specified point a local maximum, a local minimum, or neither of $f(x,y)$ subject to $g(x,y)=0$?

a)



b)



3. Find the extremal values of

$f(x,y) = xy$ where $x^3 + y^2 \le 1$.