§1. Gradients

First, the definition.

Def. Let $f(x, y, z)$ be a function of 3 variables

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ is another function of 3 variables.

$\nabla f_p = \langle f_x(p), f_y(p), f_z(p) \rangle$ for p in $\mathbb{R}^3$.

e.g. - $f(x, y, z) = xyz$

$\nabla f = \langle yz, xz, xy \rangle$

$\nabla f_{(1,1,1)} = \langle 1, 1, 1 \rangle$

- $f(x, y, z) = e^{xy} \cos(z)$

$\frac{\partial f}{\partial x} = ye^{xy} \cos(z)$, $\frac{\partial f}{\partial y} = xe^{xy} \cos(z)$, $\frac{\partial f}{\partial z} = -e^{xy} \sin(z)$

$\langle ye^{xy} \cos(z), xe^{xy} \cos(z), -e^{xy} \sin(z) \rangle$

- $f(x,y,z) = (x+y+z)^2$

$$= x^2+y^2+z^2+2xy+2xz+2yz$$

$$\nabla f = \langle 2x+2y+2z,\ 2x+2y+2z,\ 2x+2y+2z \rangle$$

Key facts

- $\nabla(f+g) = \nabla f + \nabla g$
  - $f(x,y,z) = x+y+z$

    $\nabla f = \nabla x + \nabla y + \nabla z$

    $= \langle 1,0,0\rangle + \langle 0,1,0\rangle + \langle 0,0,1\rangle$

    $= \langle 1,1,1\rangle$

- $\nabla(cf) = c\nabla(f)$ for c a constant.
  - $f(x,y,z) = 2xyz$

    $\nabla f = 2\langle yz, xz, xy\rangle$

- $\nabla(fg) = f\nabla(g) + g\nabla(f)$ $\boxed{\text{product rule}}$

- $f(x,y,z) = (x+y+z)^2$

$= (x+y+z)(x+y+z)$

$\nabla f = (x+y+z)\,\nabla(x+y+z) + (x+y+z)\,\nabla(x+y+z)$

$= 2(x+y+z)\langle 1,1,1\rangle$

$= \langle 2x+2y+2z,\ 2x+2y+2z,\ 2x+2y+2z\rangle$

- $F(t)$ a 1 variable function

$f(x,y,z)$ a 3 variable function

$\rightsquigarrow F(f(x,y,z))$ a 3 variable function

$$\nabla(F(f(x,y,z))) = \underbrace{F'(f(x,y,z))}_{\text{scalar}}\ \underbrace{\nabla f}_{\text{vector}}$$

$\boxed{\text{chain rule}}$

- $f(x, y, z) = (x+y+z)^2$

$F(t) = t^2$

$g(x, y, z) = x+y+z$

$$\nabla (F(g(x,y,z))) = F'(g(x,y,z))\, \nabla g(x,y,z)$$

$$= 2(x+y+z)\langle 1,1,1\rangle$$

$$= \langle 2x+2y+2z,\ 2x+2y+2z,\ 2x+2y+2z\rangle$$

- $f(x, y, z) = (x+yz)^{1000}$

$$\nabla f = 1000(x+yz)^{999}\, \nabla(x+yz)$$

$$= 1000(x+yz)^{999}\langle 1, z, y\rangle$$

- $f(x,y,z)$ a 3 variable function

$\vec{r}(t)$ a vector valued function in $\mathbb{R}^3$.

$f(\vec{r}(t))$ is then a scalar valued function in one variable

$\therefore \frac{d}{dt} f(\vec{r}(t))$ should be a normal single variable derivative, i.e., should be a scalar

$$\frac{d}{dt} f(\vec{r}(t)) = \underbrace{\underbrace{\nabla f_{\vec{r}(t)}}_{\text{vector}} \cdot \underbrace{\vec{r}\,'(t)}_{\text{vector}}}_{\text{scalar}}$$

— Let $f(x,y,z) = xyz$

$\vec{r}(t) = (t, t^2, t^3)$

$f(\vec{r}(t)) = t \cdot t^2 \cdot t^3 = t^6$. $\quad \frac{d}{dt} f(\vec{r}(t)) = 6t^5$

$\quad \shortparallel$

$\nabla f_{\vec{r}(t)} \cdot \vec{r}\,'(t) = \langle yz, xz, xy \rangle_{(t,t^2,t^3)} \cdot \langle 1, 2t, 3t^2 \rangle$

$$= \langle t^5, t^4, t^3 \rangle \cdot \langle 1, 2t, 3t^2 \rangle$$

$$= t^5 + 2t^5 + 3t^5$$

$$= 6t^5$$

- e.g. $f(x,y) = xy$

$$\vec{r}(t) = \langle x(t), y(t) \rangle.$$

$$\Rightarrow f(\vec{r}(t)) = x(t)\, y(t)$$

$$\frac{d}{dt}(x(t)y(t)) = \frac{d}{dt} f(\vec{r}(t))$$

$$= \nabla f_{\vec{r}(t)} \cdot \vec{r}\,'(t)$$

$\nabla f = \langle y, x \rangle$, Hence,

$$\nabla f_{\vec{r}(t)} \cdot \vec{r}\,'(t) = \langle y(t), x(t) \rangle \cdot \vec{r}\,'(t)$$

$$= \langle y(t), x(t) \rangle \cdot \langle x'(t), y'(t) \rangle$$

$$= y(t)x'(t) + x(t)y'(t)$$

Thus we proved $\frac{d}{dt}(x(t)y(t)) = y(t)x'(t) + x(t)y'(t)$,

aka the product rule!

Try this for $f(x,y) = x+y, x-y, x/y, x^y$ for more derivations, or try $f(x,y,z) = xyz$ for a formula to compute $\frac{d}{dt}(x(t)y(t)z(t))$

Moral

all chain rules look the same

derivative $(f \circ g)$ = (derivative(f) evaluated at g) times(derivative(g))

The questions are i) which form of derivative

ii) which notion of multiplication

$f(x,y,z)$, i.e., $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ has derivative ∇f, a vector

$\vec{r}(t)$, i.e. $\mathbb{R} \longrightarrow \mathbb{R}^3$, has derivative $\vec{r}\,'(t)$, a vector

$g(t)$, i.e. $\mathbb{R} \longrightarrow \mathbb{R}$, has the usual single variable calculus derivative $g'(t)$, a scalar.

$$\underbrace{\frac{d}{dt}(f \circ \vec{r})}_{\text{scalar}} = \underbrace{\nabla f_{\vec{r}(t)}}_{\text{vector}} \cdot \underbrace{\vec{r}\,'(t)}_{\text{vector}}$$

$$\underbrace{\nabla (g \circ f)}_{\text{vector}} = \underbrace{\nabla g\,(f(x,y,z))}_{\text{scalar}} \cdot \underbrace{\nabla f(x,y,z)}_{\text{vector}}$$

§2. Directional Derivatives

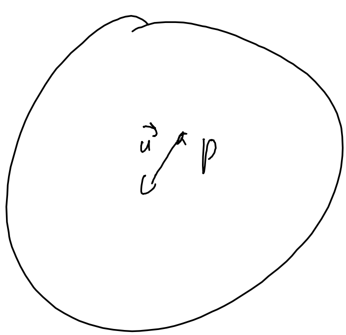
Let $\vec{u}$ be a unit vector.

Let p be a point

Let $\vec{r}(t) = p + t\vec{u}$

Let f be a function

The directional derivative of f at p in the direction $\vec{u}$ is $D_{\vec{u}} f(p) = \frac{d}{dt} f(\vec{r}(t))\Big|_{t=0}$.



e.g. $p = (a,b,c)$

$\vec{i} = (1,0,0)$

$f(x,y,z)$

$$D_{\vec{i}} f(a,b,c) = \lim_{t\to 0} \frac{f(a+t,b,c) - f(a,b,c)}{t}$$

$$= \frac{\partial f}{\partial x}(a,b,c)$$

similarly $D_{\vec{j}} f(a,b,c) = \frac{\partial f}{\partial y}(a,b,c)$

$$D_{\vec{k}} f(a,b,c) = \frac{\partial f}{\partial z}(a,b,c)$$

This is usually easiest to compute with the following

$$\boxed{D_{\vec{u}} f(p) = \nabla f_p \cdot \vec{u}}$$

$$\left(\text{pf.}\ \frac{d}{dt} f(\vec{p}(t))\Big|_{t=0} = \nabla f_{\vec{p}(0)} \cdot \vec{r}\,'(0) \right.$$
$$\left. = \nabla f_p \cdot \vec{u} \right)$$

Critical consequences

- $D_{\vec{u}} f(P) = \|\nabla f_P\| \cos(\theta)$

increase is maximized at $\theta = 0$, so in the direction of the gradient

decrease is maximized at $\theta = \pi$, so in the <u>opposite</u> direction to the gradient

e.g. $f(x,y) = xy$. What are the directions of the maximal rates of increase/decrease at $(1,1)$? at $(-1,1)$? what are the rates?

$\nabla f = \langle y, x \rangle$

$\nabla f_{(1,1)} = \langle 1, 1 \rangle$

$\vec{q} = \dfrac{\langle 1,1 \rangle}{\|\langle 1,1 \rangle\|} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ is the unit vector in the direction of $\nabla f(1,1)$.

So this is the direction of max'l increase

The rate is $\|\nabla f_{(1,1)}\| = 2$

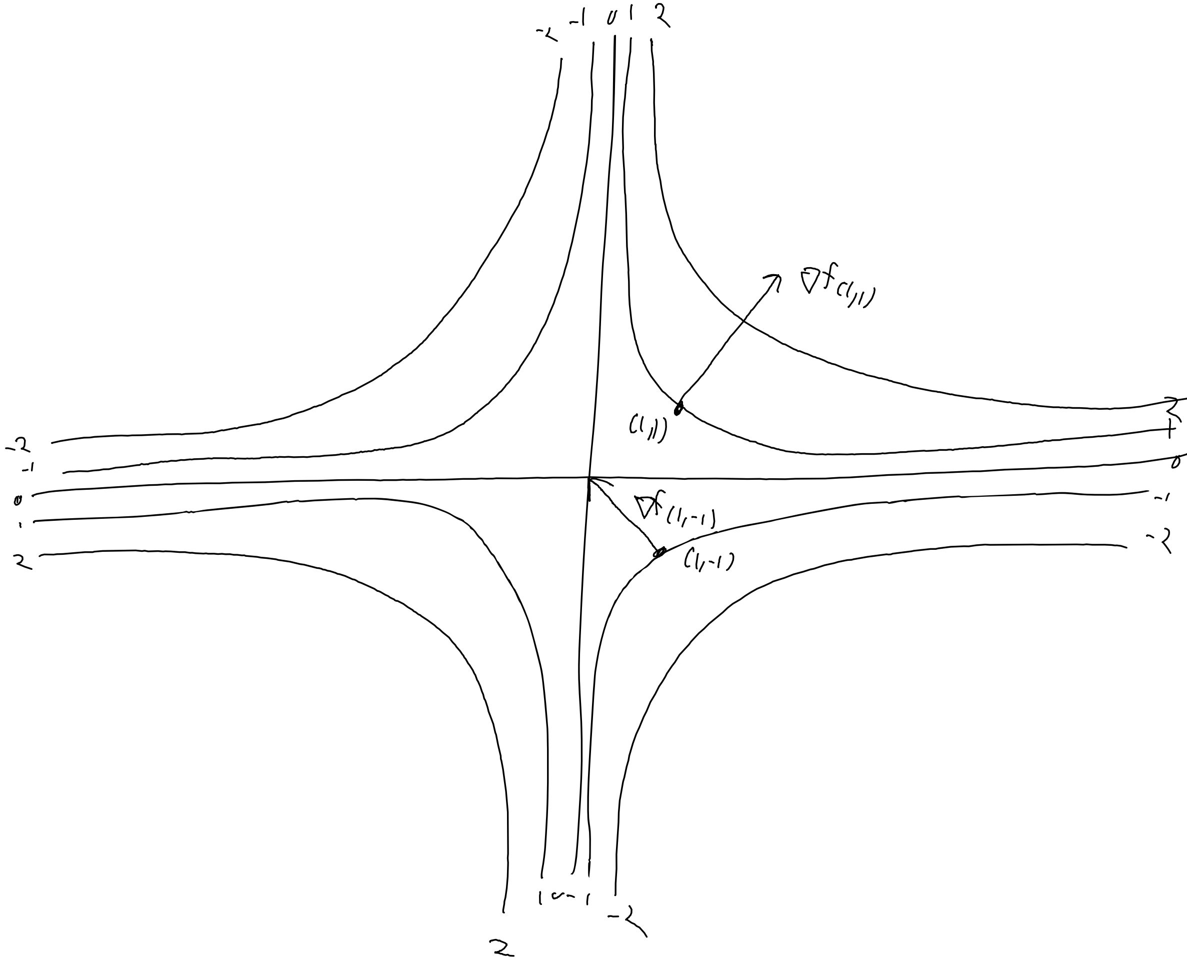
Similarly, the max'l decrease rate is in the direction $-\vec{u}$ with rate -2

At $(1,-1)$, $\nabla f_{(1,-1)} = \langle -1, 1 \rangle$

$\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$ is the direction of max'l increase with rate $\|\nabla f_{(1,-1)}\| = 2$.

$-\vec{u}$ is the direction of max'l decrease w/ rate -2.

- ∇f_p is perpendicular to the level set of f at p.



-2 -1 0 1 2
∇f(1,1)
(1,1)
∇f(1,-1)
(1,-1)
2 1 0 -1 -2