§1. Partial derivatives

Consider a function $f(x,y,z)$ of 3 variables $(f: \mathbb{R}^3 \longrightarrow \mathbb{R})$

Its partial derivatives are

$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$

$$f_z = \frac{\partial f}{\partial z} = \lim_{h \to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

Moral: In a partial derivative, all variables are constants except the one in the denominator e.g. $f(x,y,z) = xyz$

$$\frac{\partial f}{\partial x} = yz$$

$$\frac{\partial f}{\partial y} = xz$$

$$\frac{\partial y}{\partial z} = xy$$

e.g. $z = \sin(xy^2)$

$$\frac{\partial z}{\partial x} = y^2\cos(xy^2)$$

$$\frac{\partial z}{\partial y} = 2y\cos(xy^2)$$

e.g. $f(x,y,z) = \log(xz)\, e^{x+yz} - 5$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\log(xz))\, e^{x+yz} + \log(xz)\frac{\partial}{\partial x}(e^{x+yz})$$

$$= \frac{z}{xz} e^{x+yz} + \log(xz) e^{x+yz}$$

$$= \frac{1}{x} e^{x+yz} + \log(xz) e^{x+yz}$$

$$\frac{\partial f}{\partial y} = \log(xz)\, z\, e^{x+yz}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\log(xz)\right)e^{x+yz} + \log(xz)\frac{\partial}{\partial z}\left(e^{x+yz}\right)$$

$$= \frac{x}{xz}\, e^{x+yz} + \log(xz)\, y\, e^{x+yz}$$

$$= \frac{1}{z}\, e^{x+yz} + \log(xz)\, y\, e^{x+yz}$$

§2. Mixed partials

We denote $f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

$$f_{xz} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{zyx} = \frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right)$$

etc.

Critical fact. ((Clairaut's theorem on equality of mixed partials)

We have $f_{xy} = f_{yx}$, i.e. $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$

(in reasonable settings)

This holds for any variable, any number of times e.g. $f_{xyxxyyzz\,xyzxyzx}$

means to differentiate f

7 times in x

4 times in y

4 times in z

in <u>any</u> order!

Key: Remember that when differentiating in one variable (e.g. x) the other variables (e.g. y, z) are <u>constant</u>, so they differentiate easily.

e.g. $g(x,y,z,w) = x^{100}y^{1000}z^{10000}w^{100000} + \cosh\left(\dfrac{e^{xy-\log(xy)}}{\sin(x/y+y/x)}\right)$

compute g_{wxxyz}

$g_{wxxyz} = g_{xxyzw}$

$g_w = \dfrac{\partial g}{\partial w} = 100000\, x^{100}y^{1000}z^{10000}w^{99999} + 0$

Now we have 1 y, 1 z, and 2 x's left

$g_{yw} = 10^8 x^{100}y^{999}z^{10000}w^{99999}$

$g_{zyw} = 10^{12}x^{100}y^{999}z^{9999}w^{99999}$

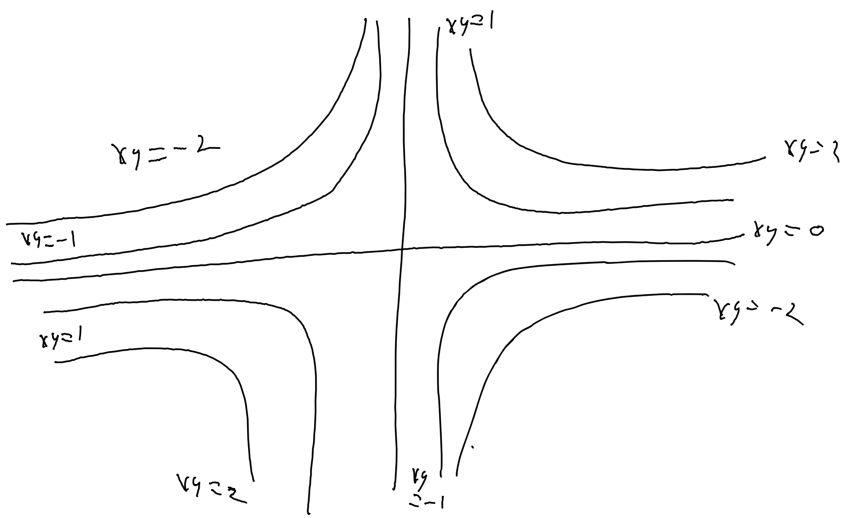
$g_{xzyw} = 10^{14}x^{99}y^{999}z^{9999}w^{99999}$

$g_{xxzyw} = 99\cdot 10^{14}x^{98}y^{999}z^{9999}w^{99999}$

§3. Contour plots

Recall for a function of two variable $f(x,y)$ that we have a contour plot by drawing the plot $C = f(x,y)$ in the xy-plane for various evenly spaced constants C.

e.g. $f(x,y) = xy$.



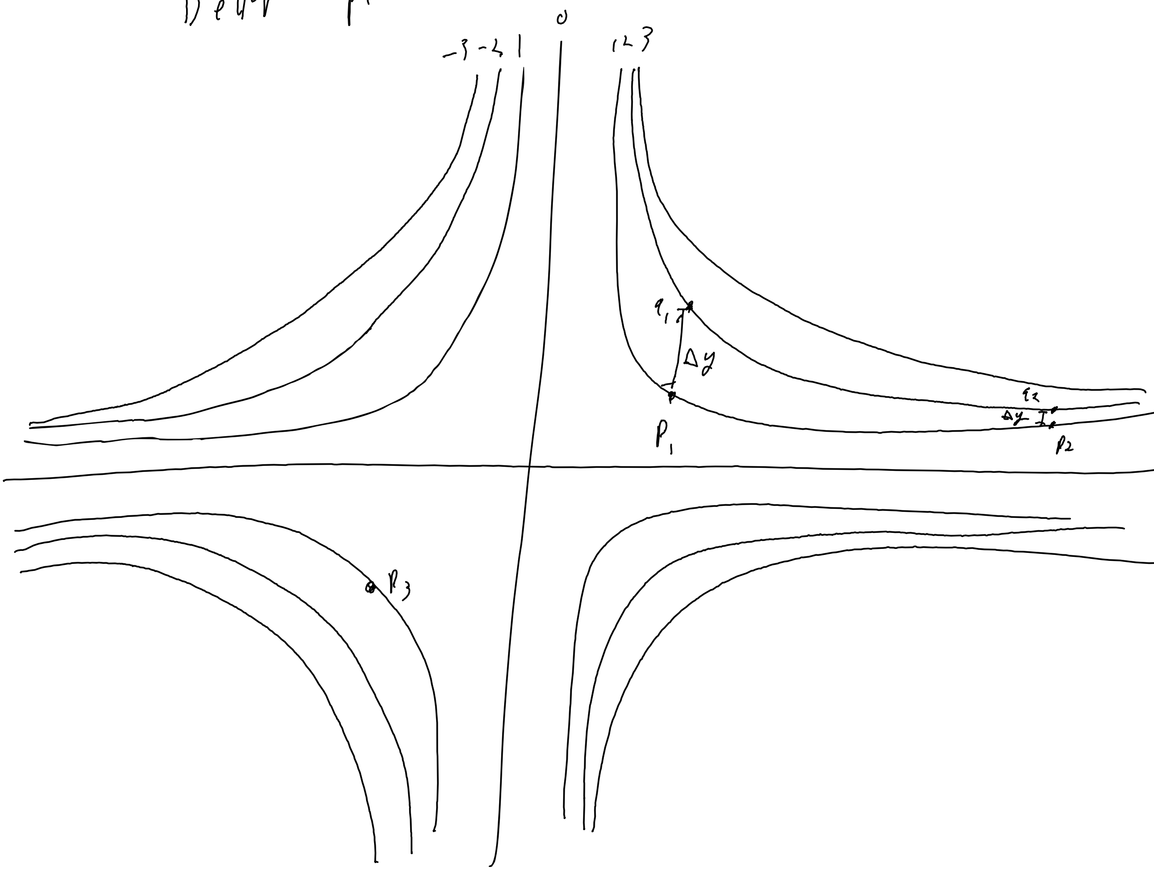
Key features:

f is constant along contour lines (think of it as altitude)

Bunched up lines mean steeper altitude

contour lines don't intersect

Better picture



Between q_1 and p_1, $\Delta z = f(q_1) - f(p_1) = 2 - 1 = 1$

Between q_2 and p_2, $\Delta z = f(q_2) - f(p_1) = 2 - 1 = 1$

Is $\dfrac{\partial z}{\partial y}$ bigger at p_1 or $\boxed{p_2}$?

↓

as the same Δz had a much smaller Δy

However, this is only the magnitude, how about sign?

Is $\frac{\partial z}{\partial y}(P_1)$ positive or negative?

we move up in y to go up in z

$\Delta z > 0$
$\Delta y > 0$ $\therefore$ $\frac{\Delta z}{\Delta y} > 0$

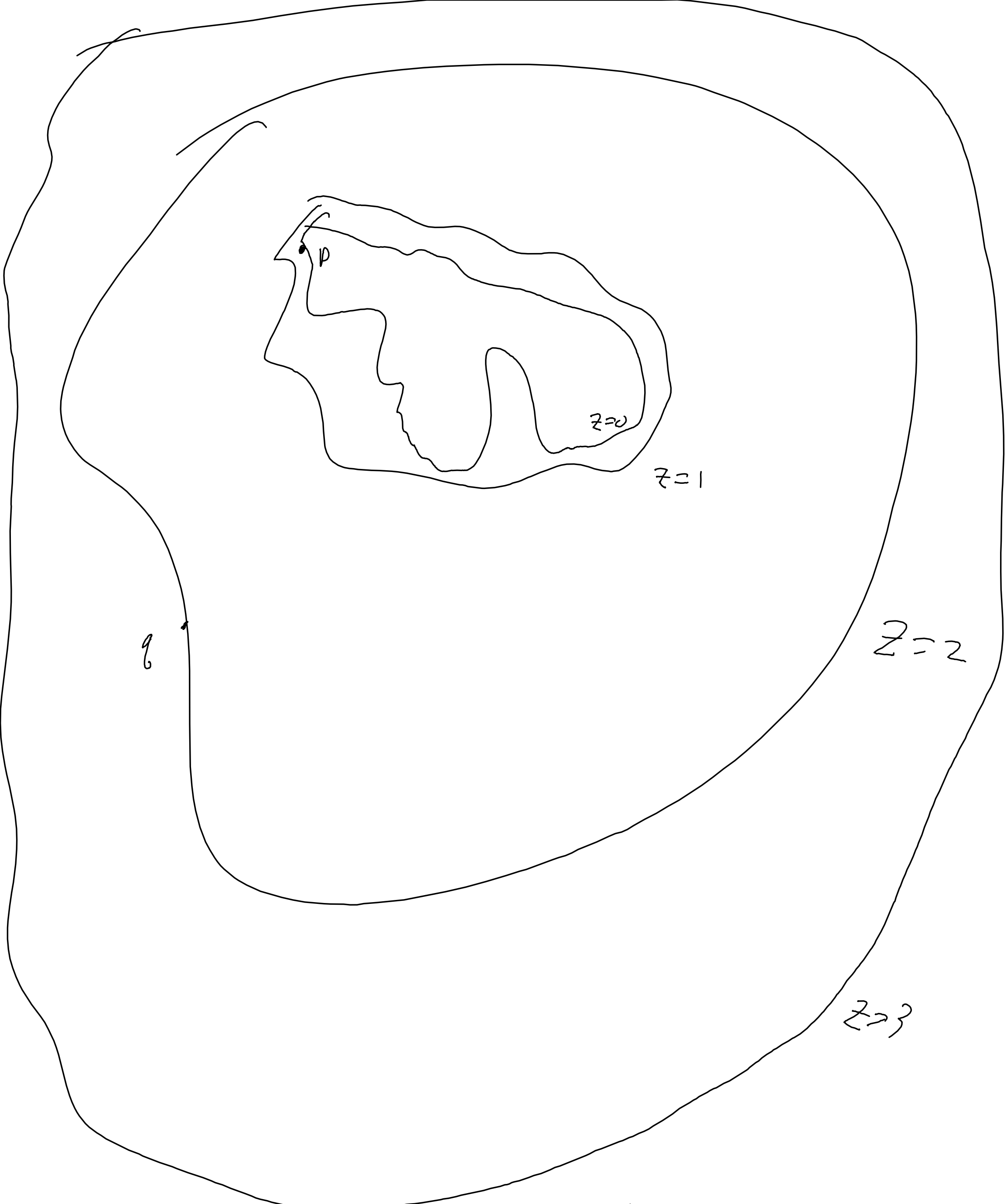
$\therefore$ $\frac{\partial z}{\partial y}(P_1) > 0$

How about @ P_2? Same thing, we go up in y to go up in z.

How about $\frac{\partial z}{\partial y}(P_3)$? Here, we go up in y to go down in z, so $\Delta z < 0$ for $\Delta y < 0$

Thus, $\frac{\Delta z}{\Delta y} < 0$

Thus, $\frac{\partial z}{\partial y}(P_3) < 0$.

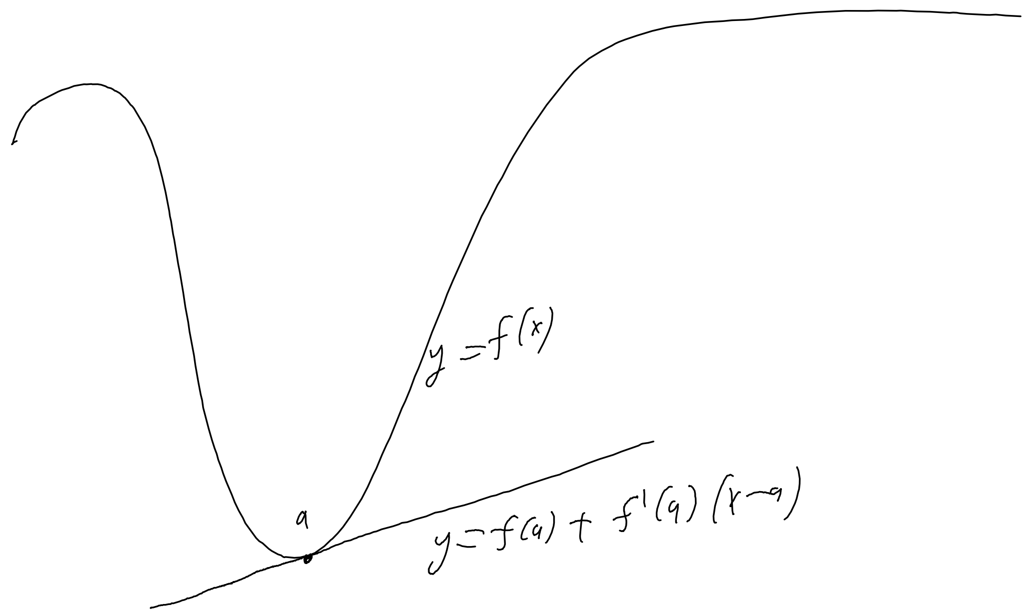


Is $\frac{\partial z}{\partial r}$ bigger at p or at q?

§ 4. Tangent planes

Recall from single variable calculus

$f(x) \approx f(a) + f'(a)(x-a)$ when $x \approx a$



Idea. $\dfrac{f(x)-f(a)}{x-a} \approx f'(a)$ when $x \approx a$

The same idea holds in higher dimensions

Say we have $f(x,y)$ and we want to linearly approximate near some (a,b)

$$\frac{f(x,b)-f(a,b)}{x-a} \approx \frac{\partial f}{\partial x}(a,b)$$

So $f(x,b) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a)$

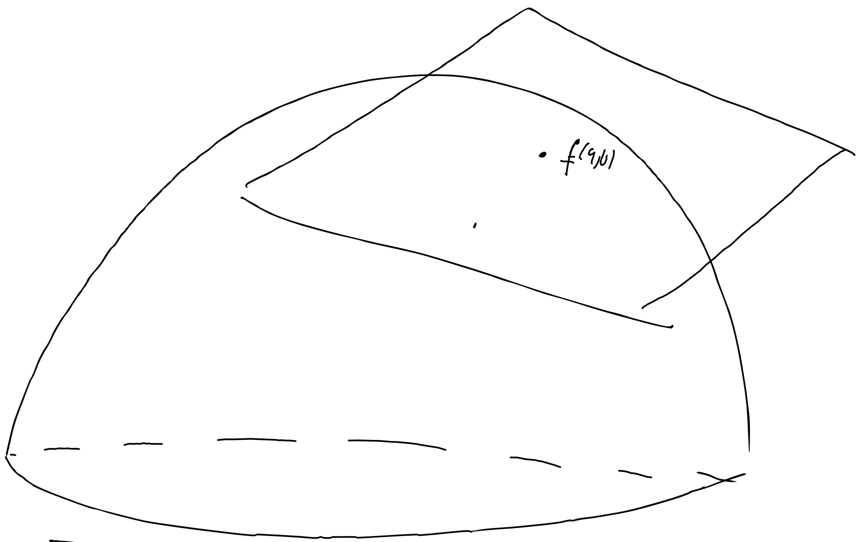
$$\frac{f(x,y)-f(x,b)}{y-b} \approx \frac{\partial f}{\partial y}(x,b) \approx \frac{\partial f}{\partial y}(a,b)$$

So $f(x,y) \approx f(x,b) + \frac{\partial f}{\partial y}(a,b)(y-b)$

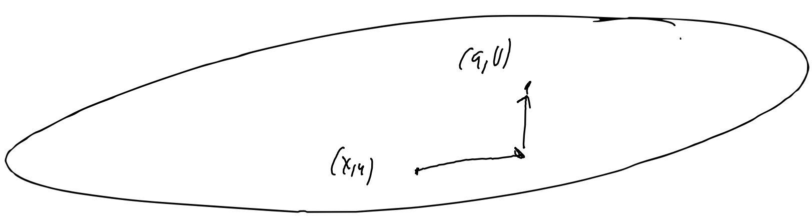
$$\approx \underbrace{f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(y-b)}_{\parallel}$$

$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$z = f(x,y)$ is thus approximated by the tangent plane $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(z-b)$



$z = \sqrt{x^2+y^2}$



e.g. - let $f(x,y) = e^{x/y}$

Find the tangent plane at $(2,1)$

$\frac{\partial f}{\partial x} = \frac{1}{y} e^{x/y}$, $\frac{\partial f}{\partial x}(2,1) = e^2$

$\frac{\partial f}{\partial y} = \frac{-x}{y^2} e^{x/y}$, $\frac{\partial f}{\partial y}(2,1) = -2e^2$

$f(2,1) = e^2$

$f(x,y) \doteq e^2 + e^2(x-2) - 2e^2(y-1)$ is the linear approximation

$z = e^2 + e^2(x-2) - 2e^2(y-1)$ is the tangent plane

- When are the tangent planes to $z = 3x^2 - xy - y^2$ parallel to the plane $\{x+y+z=0\}$

The tangent plane at (a,b) is

$$z = f(a,b) + \frac{\partial z}{\partial x}(a,b)(x-a) + \frac{\partial z}{\partial y}(a,b)(y-b)$$

So its normal vector is $\left\langle \frac{\partial z}{\partial x}(a,b), \frac{\partial z}{\partial y}(a,b), -1 \right\rangle$.

To be parallel to the plane $\{x+y+z=0\}$, w/ normal vector $\langle 1,1,1 \rangle$, means that $\left\langle \frac{\partial z}{\partial x}(a,b), \frac{\partial z}{\partial y}(a,b), -1 \right\rangle$ and $\langle 1,1,1 \rangle$ must be parallel.

So we need

$$\frac{\partial z}{\partial x}(a,b) = -1$$

$$\frac{\partial z}{\partial y}(a,b) = -1$$

Now, $\frac{\partial z}{\partial x} = 6x - y$

$\frac{\partial z}{\partial y} = -x - 2y$

we need $6x - y = -1 \rightarrow y = 6x+1$

$-x - 2y = -1 \rightarrow -x - 2(6x+1) = -1$

$-x - 12x - 2 = -1$

$-13x = 1$

$x = \frac{-1}{13}$, $y = \frac{7}{13}$

So the tangent plane to $z = 3x^2 - xy - y^2$ is parallel to the plane $\{x+y+z=0\}$ only at the point $\left(\frac{-1}{13}, \frac{7}{13}\right)$.