Midterm 1 Solutions

1) a) $\vec{r}_1(t) = (-2,-5,1) + t\langle 3,4,1\rangle$

$= (-2+3t, -5+4t, 1+t)$

b) Let $\vec{r}_1(t_1) = \vec{r}_2(t_2)$

$(-2+3t_1, -5+4t_1, 1+t_1) = (-1-t_2, -3-t_2, 8+3t_2)$

Thus,

$$-2+3t_1 = -1-t_2 \qquad (1)$$
$$-5+4t_1 = -3-t_2 \qquad (2)$$
$$1+t_1 = 8+3t_2 \qquad (3)$$

By (1), $t_2 = 1-3t_1$

plug this into (2) to get

$$-5+4t_1 = -3-(1-3t_1)$$
$$= -4+3t_1$$

so

$$-5+t_1 = -4$$
$$t_1 = 1$$

plugging this back into $\lrcorner$ yields $t_2 = -2$

plug $t_1 = 1$, $t_2 = -2$ into (3) to get

$$1+1 = 8-6$$

which is true, so $\vec{r}_1(1) = \vec{r}_2(-2) = (1,-1,2)$.

2) $\vec{PQ} = \langle 2-1, 4-1, 3-2 \rangle = \langle 1, 3, 1 \rangle$

$\vec{PR} = \langle 3-1, 3-1, -1-2 \rangle = \langle 2, 2, -3 \rangle$

Let $\vec{n} = \vec{PQ} \times \vec{PR}$

$$= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 1 \\ 2 & 2 & -3 \end{pmatrix}$$

$$= \vec{i} \det \begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix} - \vec{j} \det \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} + \vec{k} \det \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

$$= -11\vec{i} + 5\vec{j} - 4\vec{k}$$

$$= \langle -11, 5, -4 \rangle$$

An equation for the plane is thus

$$-11(x-1) + 5(y-1) - 4(z-2) = 0$$

or $\quad -11x + 5y - 4z = -11 + 5 - 8$

$$= -14$$

3) a) $\vec{r}'(t) = \langle \frac{d}{dt}(4t), \frac{d}{dt}(\cos(3t)), \frac{d}{dt}(\sin(3t)) \rangle$

$= \langle 4, -3\sin(3t), 3\cos(3t) \rangle$

$\vec{r}''(t) = \langle 0, -9\cos(3t), -9\sin(3t) \rangle$

b) We begin our measurement at $t=0$, which is the point $\vec{r}(0) = \langle 0, 1, 0 \rangle$.

1) Let $s = g(t) = \int_0^t \|\vec{r}'(u)\| \, du$

$= \int_0^t \sqrt{4^2 + (-3\sin(3u))^2 + (3\cos(3u))^2} \, du$

$= \int_0^t \sqrt{16 + 9(\sin(3u)^2 + \cos(3u)^2)} \, du$

$= \int_0^t \sqrt{16+9} \, du$

$= \int_0^t 5 \, du$

$= 5t$

2) Solve $t = g^{-1}(s)$. Here, $t = \frac{s}{5}$

3) plug in $t = g^{-1}(s)$ to $\vec{r}(t)$.
$\vec{r}_1(s) = \vec{r}(g^{-1}(s)) = \langle \frac{4}{5}s, \cos(3s), \sin(3s) \rangle$.

4) a) NOTE: The correct formula is

$$\boxed{\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}}$$

$$\vec{r}\,'(t) = \langle 1+3t^2, -\sin(t), e^t \rangle$$

$$\vec{r}\,''(t) = \langle 6t, -\cos(t), e^t \rangle$$

$$\kappa(0) = \frac{\|\vec{r}\,'(0) \times \vec{r}\,''(0)\|}{\|\vec{r}\,'(0)\|^3}$$

$$= \frac{\|\langle 1,0,1\rangle \times \langle 0,-1,1\rangle\|}{\|\langle 1,0,1\rangle\|^3}$$

$$\langle 1,0,1\rangle \times \langle 0,-1,1\rangle = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \vec{i}\det\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} - \vec{j}\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$+ \vec{k}\det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \vec{i} - \vec{j} - \vec{k}$$

$$= \langle 1, -1, -1 \rangle$$

$\|\langle 1,0,1\rangle\|^3 = \sqrt{8}$, $\|\langle 1,-1,-1\rangle\| = \sqrt{3}$

$$\therefore \kappa(0) = \sqrt{\frac{3}{8}}$$

b) The radius of the osculating circle is reciprocal to the curvature, so the radius is $\sqrt{\frac{8}{3}}$.

Arc Length Parametrization

Crucial point: Computing $s = g(t) = \int_{t_0}^{t} \|\vec{r}\,'(u)\|\,du$

is not enough! You must also invert this relationship to write $t = g^{-1}(s)$ and plug it into $\vec{r}(t)$

$$\vec{r}_1(s) = \vec{r}(g^{-1}(s))$$

is the arc length parametrization of $\vec{r}(t)$.

$$\frac{d\vec{r}_1}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds}$$ by the chain rule

If you prefer,

$$\vec{r}_1\,'(s) = \vec{r}\,'(g^{-1}(s))\,(g^{-1})'(s)$$

We have

$$\frac{dt}{ds} = \frac{1}{\left(\frac{ds}{dt}\right)}$$

and $$s = \int_{t_0}^{t} \|\vec{r}\,'(u)\|\,du$$

so $$\frac{ds}{dt} = \|\vec{r}\,'(t)\|$$ by FTC

so $$\frac{1}{\left(\frac{dt}{ds}\right)} = \frac{1}{\|\vec{r}\,'(t)\|}$$

Hence, $\frac{d\vec{r}_1}{ds} = \frac{d\vec{r}}{dt}\,\frac{1}{\|r'(t)\|}$

so $\left\| \frac{d\vec{r}_1}{ds} \right\| = \|\vec{r}\,'(t)\| \, \frac{1}{\|\vec{r}\,'(t)\|} = 1$

Examples

- Which of the following is an arc length parametrization of a circle of radius 4 centered at $(0,0)$?

$\vec{r}_1(t) = \langle 4\sin(t), 4\cos(t) \rangle$

$\vec{r}_2(t) = \langle 4\sin(4t), 4\cos(4t) \rangle$

$\boxed{\vec{r}_3(t) = \langle 4\sin(\tfrac{1}{4}t), 4\cos(\tfrac{1}{4}t) \rangle}$

- Find an arclength parametrization for $\vec{r}(t) = \langle t^2, t^3 \rangle$ for $t \geq 0$.

Step 0. No initial pt specified, so take $t_0 = 0$ and $\vec{r}(t_0) = 0$.

Step 1. Evaluate $s = g(t) = \int_0^t \|\vec{r}\,'(u)\|\,du$.

$\vec{r}\,'(u) = \langle 2u, 3u^2 \rangle$

$\|\vec{r}\,'(u)\| = \sqrt{4u^2 + 9u^4} = u\sqrt{4 + 9u^2}$

We must evaluate

$$\int_0^t u\sqrt{4+9u^2}\, du$$

Let $v = 4+9u^2$, so $dv = 18u\, du$, so $u\,du = \frac{1}{18}dv$

$$\int_{u=0}^{u=t} u\sqrt{4+9u^2}\, du = \int_{v=4}^{v=4+9t^2} \frac{1}{18}\sqrt{v}\, dv$$

$$= \frac{1}{18}\left(\frac{2}{3}v^{3/2}\right)\Bigg|_{v=4}^{v=4+9t^2}$$

$$= \frac{1}{27}\left((4+9t^2)^{3/2} - 8\right)$$

So $\quad s = \frac{1}{27}\left((4+9t^2)^{3/2} - 8\right)$

Step 2, Solve for t

$$27s + 8 = (4+9t^2)^{3/2}$$

$$\sqrt{\frac{(27s+8)^{2/3} - 4}{9}} = t \quad (\text{recall } t \geq 0)$$

Step 3, plug this into $\vec{r}$

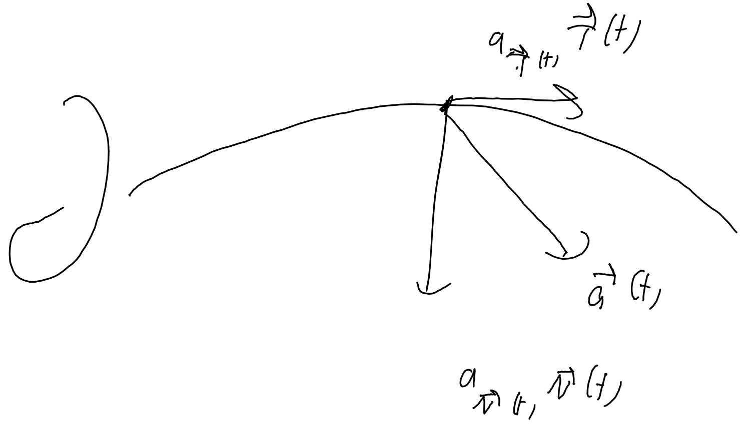
$$\vec{r}_1(s) = \left\langle \frac{(27s+8)^{2/3}-4}{9},\ \left(\frac{(27s+8)^{2/3}-4}{9}\right)^{3/2} \right\rangle$$

Acceleration, tangential and normal

Recall $\vec{a}(t) = \vec{r}''(t)$.

As a vector, it has projections onto other vectors. We are especially interested in $\vec{T}(t)$ and $\vec{N}(t)$. We have then a decomposition

$$\vec{a}(t) = a_{\vec{T}(t)}\vec{T}(t) + a_{\vec{N}(t)}\vec{N}(t)$$

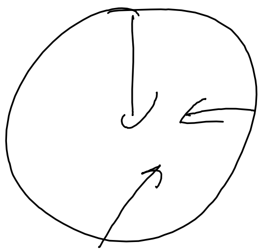


Fact. $a_{\vec{T}(t)} = v'(t)$, $v(t) = \|\vec{r}'(t)\|$

$a_{\vec{N}(t)} = \kappa(t)\, v(t)^2$

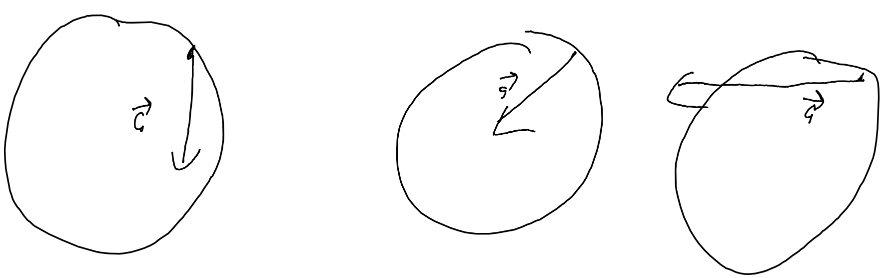
e.g., uniform circular motion

$$\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle$$



acceleration is solely in the normal direction

Consider the following acceleration vectors in a non-uniform circular motion



Is the speed increasing, decreasing, or constant at these points?

e.g., Let $\vec{r}(t) = \langle t^2, 2t, \ln(t) \rangle$

Find the decomposition of its acceleration vector at $t = \frac{1}{2}$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, 2, \tfrac{1}{t} \rangle$$

$$\vec{a}(t) = \vec{v}\,'(t) = \langle 2, 0, \tfrac{-1}{t^2} \rangle$$

So $\vec{v}(1/2) = \langle 1, 2, 2 \rangle$

$$\vec{a}(1/2) = \langle 2, 0, -4 \rangle$$

Hence, $$\vec{T}(1/2) = \frac{\vec{v}(1/2)}{\|\vec{v}(1/2)\|} = \frac{\langle 1, 2, 2 \rangle}{\sqrt{1^2+2^2+2^2}} = \langle \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3} \rangle$$

Thus, $$a_{\vec{T}}(1/2) = \vec{a}(1/2) \cdot \vec{T}(1/2)$$
$$= \langle 2, 0, -4 \rangle \cdot \langle \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3} \rangle$$
$$= -2$$

and $$a_{\vec{N}}(1/2)\,\vec{N}(1/2) = \vec{a}(1/2) - a_{\vec{T}}(1/2)\,\vec{T}(1/2)$$
$$= \langle 2, 0, -4 \rangle - (-2)\langle \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3} \rangle$$
$$= \langle \tfrac{8}{3}, \tfrac{4}{3}, -\tfrac{8}{3} \rangle$$

Then $\vec{N}(1/2)$ is the associated unit vector

$$a_{\vec{N}} = \|a_{\vec{N}}(1/2)\vec{N}\| = \sqrt{\left(\frac{8}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(-\frac{8}{3}\right)^2}$$

$$= 4$$

So $$\vec{N}(1/2) = \frac{\langle \frac{8}{3}, \frac{4}{3}, -\frac{8}{3} \rangle}{4} = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle$$

Finally, $$\vec{a}(1/2) = a_{\vec{T}(1/2)}\vec{T}(1/2) + a_{\vec{N}(1/2)}\vec{N}(1/2)$$

$$= -2\vec{T}(1/2) + 4\vec{N}(1/2).$$