Arc Length

Notation. t is the time parameter

s is the arclength parameter

Let $\vec{r}(t)$ be some parametrized curve (in $\mathbb{R}^2$, $\mathbb{R}^3$, whatever).

The distance from $t=a$ to $t=b$ along $\vec{r}$ is

$$s = \int_a^b \|r'(t)\| \, dt$$

If we fix a start time $t=a$, we get a function

$$s(t) = \int_a^t \|r'(u)\| \, du$$

e.g. what is the speed of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at time t?

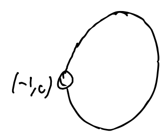
$$s'(t) = \|r'(t)\| \text{ by FTC}$$

$$= \|\langle 1, 2t, 3t^2 \rangle\| = \sqrt{1 + 4t^2 + 9t^4}.$$

eg. $\vec{r}(t) = \left\langle \dfrac{1-t^2}{1+t^2}, \dfrac{2t}{1+t^2} \right\rangle$, $t \in (-\infty, \infty)$.

Compute the arc length of $\vec{r}(t)$

(Note. $\vec{r}(t)$ parametrizes the unit circle except for $(-1,0)$)



$$\vec{r}\,'(t) = \left\langle \frac{(-2t)(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}, \frac{2(1+t^2) - (2t)^2}{(1+t^2)^2} \right\rangle$$

$$= \frac{1}{(1+t^2)^2} \left\langle -2t(2t^2), 2+2t^2-4t^2 \right\rangle$$

$$= \frac{1}{(1+t^2)^2} \left\langle -4t^3, 2-2t^2 \right\rangle$$

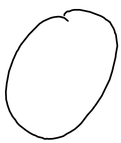
$$\|\vec{r}\,'(t)\|^2 = \frac{1}{(1+t^2)^4} \left(16t^6 + (2-2t^2)^2 \right)$$

$$= \frac{1}{(1+t^2)^4} \left(16t^6 + 4t^4 - 4t^2 + 2 \right)$$

(incomplete)

Arc length parametrization

The same curve can be parametrized in countless ways



$\langle \cos(t), \sin(t) \rangle$

$\langle \cos(t(t-1)(t+1)), \sin(t(t-1)(t+1)) \rangle$

$\vdots$

An arc length parametrization $\vec{r}(s)$ of a curve is one w/ constant unit speed.

$$\|\vec{r}\,'(s)\| = 1 \quad \text{for all } s$$

$\vec{r}(t)$ is the pt on the curve after t seconds.

$\vec{r}(s)$ is the pt on the curve s meters from the start.

e.g. Find an arc length parametrization of

$\vec{r}(t) = \langle 3t+1,\ 4t-5,\ 2t \rangle$ starting from $(1,-5,0)$

0. Find t s.t. $\vec{r}(t) = (1,-5,0)$, $t=0$.

1. Compute $s = g(t)$ the arc length integral

$$g(t) = \int_0^t \|\vec{r}\,'(u)\|\, du$$

$$\vec{r}\,'(u) = \langle 3, 4, 2 \rangle$$

$$\|\vec{r}\,'(u)\| = \sqrt{9+16+4}$$

$$= \sqrt{29}$$

so $g(t) = \int_0^t \sqrt{29}\, du$

$$= \sqrt{29}\, t$$

2. Compute $t = g^{-1}(s)$

$s = \sqrt{29}\, t$ so $t = \frac{s}{\sqrt{29}}$, i.e. $g^{-1}(s) = \frac{s}{\sqrt{29}}$

3. $\vec{r}_1(s) = \vec{r}(g^{-1}(s))$ is the arc length parametrization

$$\vec{r}_1(s) = \left\langle \frac{3}{\sqrt{29}} s + 1,\ \frac{4}{\sqrt{29}} s - 5,\ \frac{2}{\sqrt{29}} s \right\rangle$$

e.g. find arc length parametrization of

$\vec{r}(t) = \langle \cos(t), \sin(t), \frac{2}{3} t^{3/2} \rangle$ starting at $(1,0,0)$

0. $\vec{r}(0) = (1,0,0)$

1. $s = g(t) = \int_0^t \|\vec{r}\,'(u)\| \, du$

$\|\vec{r}\,'(u)\| = \| \langle -\sin(u), \cos(u), t^{1/2} \rangle \|$

$= \sqrt{(-\sin(u))^2 + (\cos(u))^2 + u}$

$= \sqrt{1 + u}$

$g(t) = \int_0^t \sqrt{1+u} \; du$

$\frac{d}{du} \frac{2}{3} (1+u)^{3/2} = \sqrt{1+u}$

so by FTC, $g(t) = \frac{2}{3} (1+u)^{3/2} \Big|_{u=0}^{u=t}$

$= \frac{2}{3} (1+t)^{3/2}$

2. Compute $s = g^{-1}(t)$

$$s = \frac{2}{3}(1+t)^{3/2}$$

$$\frac{3}{2}s = (1+t)^{3/2}$$

$$\left(\frac{3}{2}s\right)^{2/3} = 1+t$$

$$\left(\frac{3}{2}s\right)^{2/3} - 1 = t$$

$$t = g^{-1}(s) = \left(\frac{3}{2}s\right)^{2/3} - 1$$

3. Compose $\vec{r}_1(s) = \vec{r}(g^{-1}(s))$

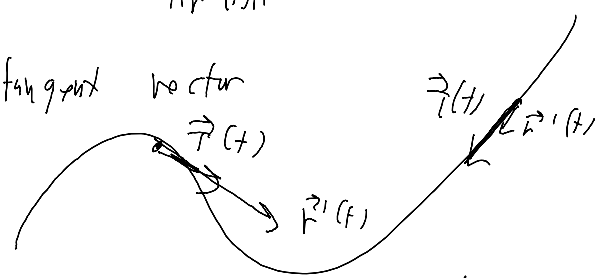
$$= \left\langle \cos\left(\left(\frac{3}{2}s\right)^{2/3} - 1\right), \sin\left(\left(\frac{3}{2}s\right)^{2/3} - 1\right), \frac{2}{3}\left(\left(\frac{3}{2}s\right)^{2/3} - 1\right)^{3/2} \right\rangle.$$

Curvature

Let $\vec{r}(t)$ be a regular parametrized curve

$(\vec{r}\,'(t) \neq \vec{0}$ for all $t)$

Then $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ makes sense. This is the unit tangent vector



curvature

Theoretical definition. $\kappa(s) = \left\| \dfrac{d\vec{T}}{ds} \right\|$

This is really hard to compute in general...

e.g. - If $\vec{r}(t)$ is a line, $\vec{T}(t)$ is constant so $\kappa(s) = 0$ for all s

- If $\vec{r}(t)$ is a circle of radius R, $\kappa(s) = 1/R$ for all s via finding an arc length parametrization.

e.g. Say $T'(s) = \langle 1, 2, 3 \rangle$ for an arc length parametrization $\vec{r}(s)$. What is $\kappa(s)$ here?

$$= \|T'(s)\| = \sqrt{1+4+9} = \sqrt{14}$$

We can also find κ in terms of t

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

$$= \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

often easier, skips a product rule as in $\|\vec{T}'(t)\|$

Rmk. If $\vec{r}(t) = (x(t), y(t))$, the 2^{nd} formula works by plugging in $(x(t), y(t), 0)$.

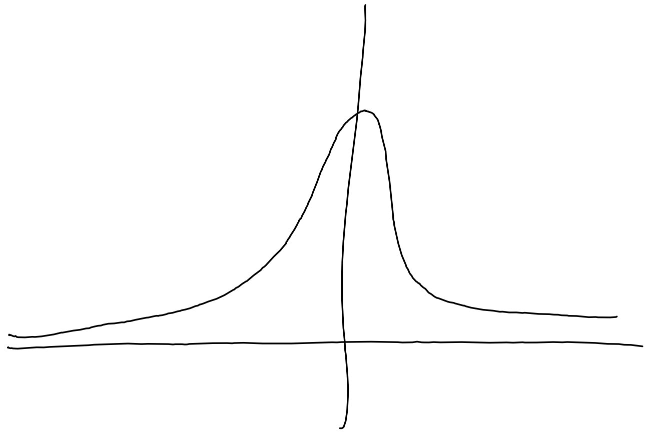
e.g. $\vec{r}(t) = (t, t^2, t^3)$. Find $\kappa(t)$

$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$, hence regular.

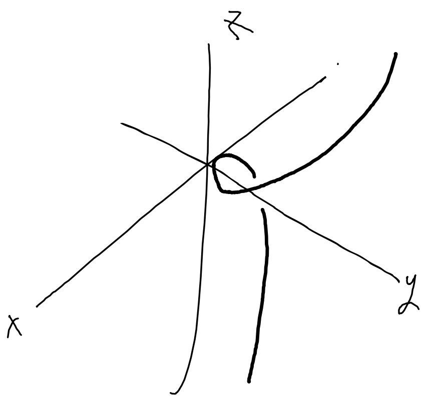
$\vec{r}''(t) = \langle 0, 2, 6t \rangle$

$$\vec{r}'(t) \times \vec{r}''(t) = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2 \vec{i} - 6t \vec{j} + 2\vec{k}$$

Thus, $$\kappa(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^3)^{3/2}}$$



$u(t)$



$\vec{r}(t)$

Oscalating planes/ circles

Recall for $\vec{r}(t)$ (regular) we had $\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

We have $\vec{T}\,'(t) \cdot \vec{T}(t) = 0$

(Pf. $\vec{T}(t) \cdot \vec{T}(t) = 1$ constant, so its derivative is 0.
on the other hand, the (dot) product rule says
$\frac{d}{dt}\left(\vec{T}(t) \cdot \vec{T}(t)\right) = \vec{T}\,'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}\,'(t) = 2\vec{T}\,'(t) \cdot \vec{T}(t)$)

That is, $\vec{T}\,'(t)$ is perpendicular to $\vec{r}(t)$ at t.

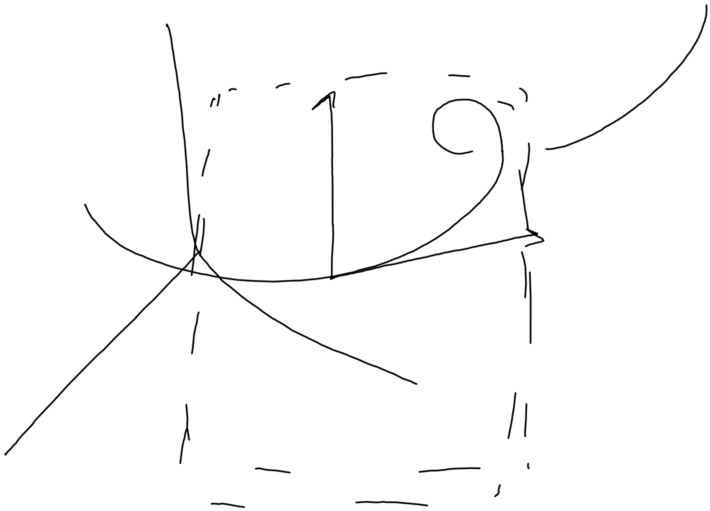
So let $\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$, the unit normal vector

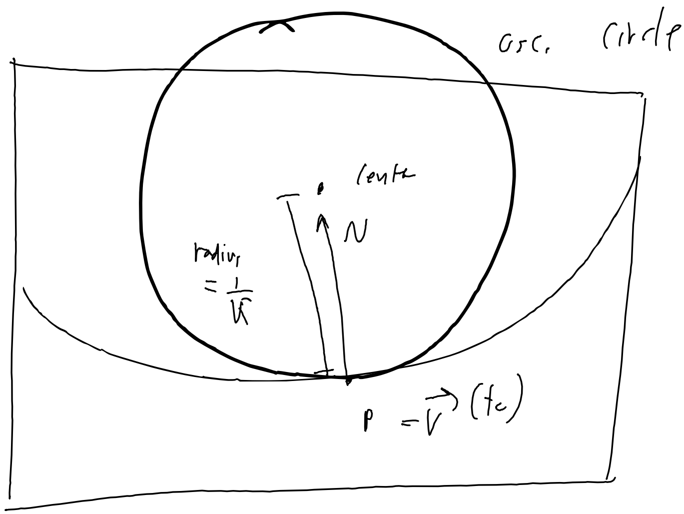
Def. The oscalating plane to a regular parametrized curve $\vec{r}(t)$ at a point $p = \vec{r}(t_0)$ is the plane passing through $p = \vec{r}(t_0)$ along the two vectors $\vec{T}(t_0)$, $\vec{N}(t_0)$

what is a normal vector to the oscalating plane?

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Then $\{\vec{T}, \vec{N}, \vec{B}\}$ is a right handed system called the Frénet frame





Center: $\vec{r}(t_0) + \frac{1}{K(t_0)} \vec{N}(t_0)$

radius: $\frac{1}{K(t_0)}$

e.g. Find the osculating circle to $\vec{r}(t) = (t, t^2, t^3)$

at $t=0$.

1. Find the osculating plane

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$T(t) = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1+4t^2+9t^4}}, \quad \vec{T}(0) = \langle 1, 0, 0 \rangle$$

$$\vec{N}(0) = \frac{\vec{T}'(0)}{\|\vec{T}'(0)\|}$$

$$\vec{T}'(t) = \underbrace{\frac{d}{dt}\left(\frac{1}{\sqrt{1+4t^2+9t^4}}\right)}\langle 1, 2t, 3t^2 \rangle + \frac{1}{\sqrt{1+4t^2+9t^4}}\langle 0, 2, 6t \rangle$$

$$\vec{T}'(0) = \perp\Big|_{t=0} \cdot \langle 1, 0, 0 \rangle + \langle 0, 2, 0 \rangle$$

$$\frac{d}{dt}\frac{1}{\sqrt{1+4t^2+9t^4}} = -\frac{3}{2}\left(1+4t^2+9t^4\right)^{-5/2} \cdot \left(8t+36t^3\right)$$

at $t=0$, get 0

so $\vec{T}'(0) = \langle 0, 2, 0 \rangle$, thus, $\vec{N}(0) = \langle 0, 1, 0 \rangle$.

So the osculating plane is the xy-plane

2. Find $\kappa(0)$, radius

we have $\kappa(t) = \dfrac{\sqrt{36t^4 + 36t^2 + 4}}{(1+4t^2+9t^3)^{3/2}}$ from before.

So $\kappa(0) = 2$

Thus the radius is $1/2$

3. Find center

$\underset{P}{(0,0,0)} + \underset{\text{radius}}{\frac{1}{2}} \underset{N}{\langle 0,1,0 \rangle} = (0, \frac{1}{2}, 0)$

Thus, the circle is centered at $(0, \frac{1}{2}, 0)$ w/ radius $1/2$.

$\{ x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2, \underbrace{z = 0}_{\text{eq'n for osculating plane}} \}$

$(\frac{1}{2}\cos(t), \frac{1}{2}(\sin(t) - \frac{1}{2}), 0)$