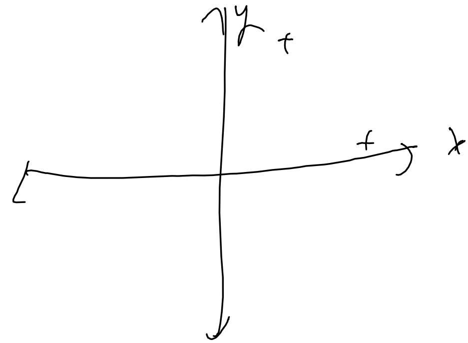
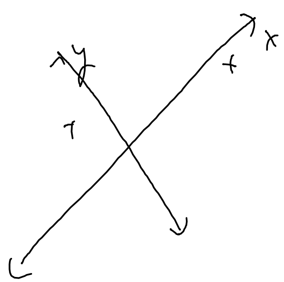
Right Handedness

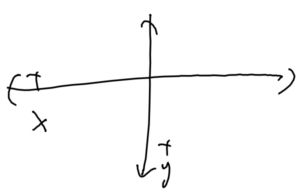
Consider the usual axes in $\mathbb{R}^2$



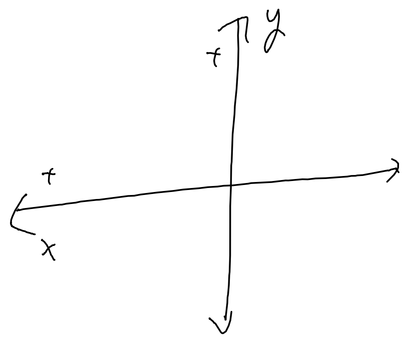
Here the positive x-axis is to the right of the positive y-axis.

If I rotate the axes, this $\boxed{\text{orientation}}$ is unchanged.



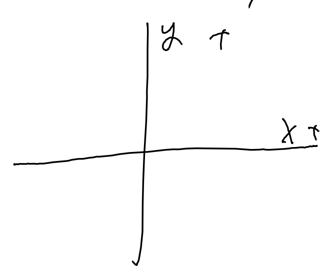


However, if I <u>reflect</u> the axes, this orientation flips.



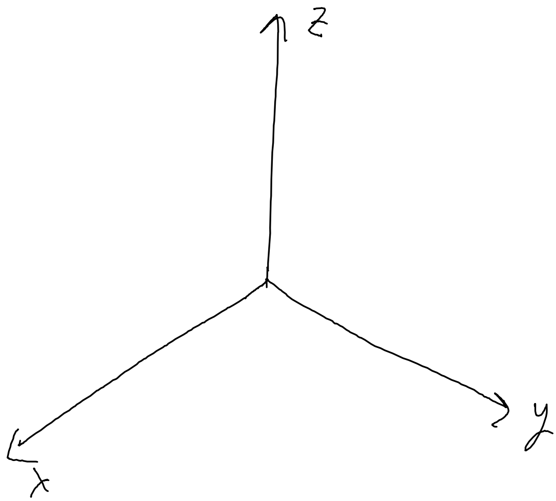
Now the positive x-axis is left of the positive y-axis.

The choice between the two is arbitrary, but we will choose the first one.



This motivates the situation in $\mathbb{R}^3$.

The standard axes are

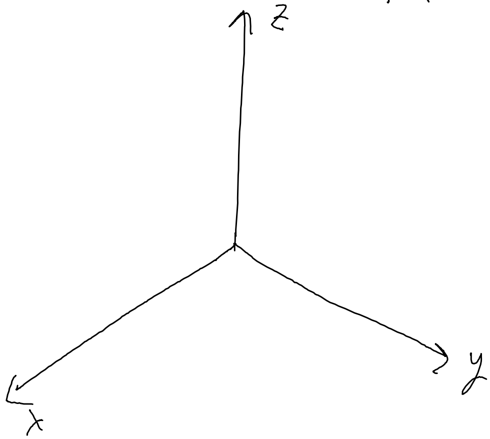


where the rays drawn are the positive directions.

This is a <u>right-handed</u> system.

Examples

R



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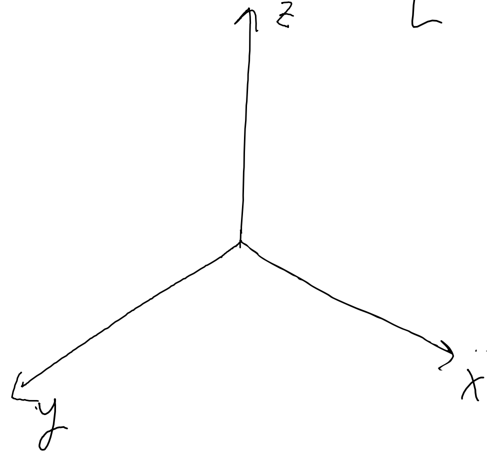

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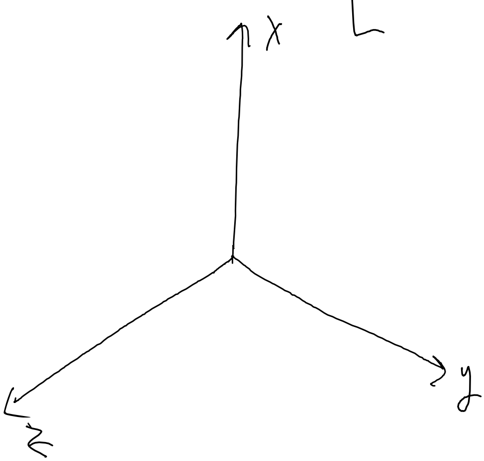

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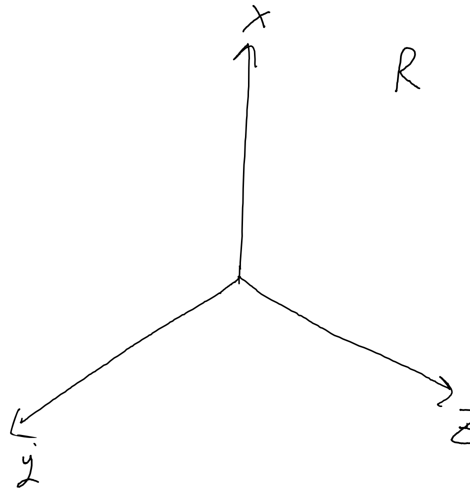

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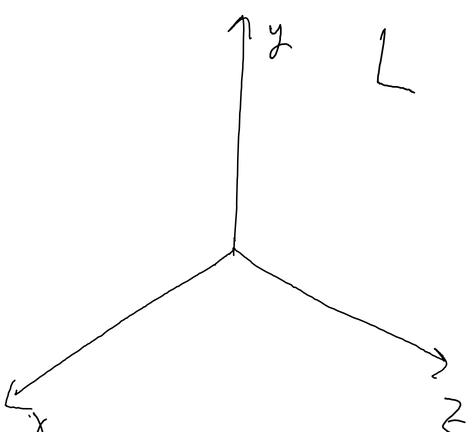

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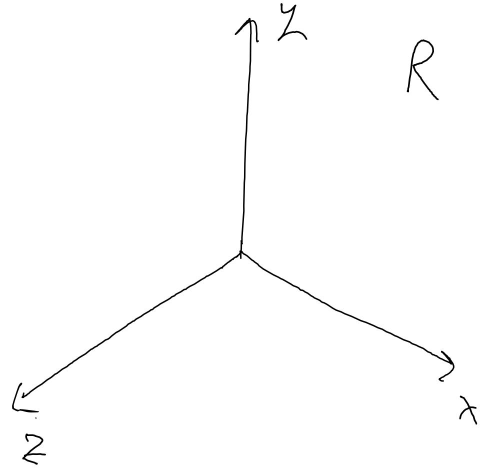


Cross products

First, an example

$$\langle 1, 2, 3 \rangle \times \langle -1, 0, 5 \rangle$$

Method 1. $\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & 0 & 5 \end{pmatrix} = \vec{i} \det \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix} - \vec{j} \det \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} + \vec{k} \det \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$

$$= 10\vec{i} - 8\vec{j} + 2\vec{k}$$

$$= \langle 10, -8, 2 \rangle$$

Method 2. $(\vec{i} + 2\vec{j} + 3\vec{k}) \times (-\vec{i} + 5\vec{k})$

$$= \left(\cancel{-\vec{i} \times \vec{i}} - 2\vec{j} \times \vec{i} - 3\vec{k} \times \vec{i}\right)$$

$$+ \left(5\vec{i} \times \vec{k} + 10\vec{j} \times \vec{k} + \cancel{15\vec{k} \times \vec{k}}\right)$$

$$= 2\vec{k} - 3\vec{j} - 5\vec{j} + 10\vec{i}$$

$$= \langle 10, -8, 2 \rangle$$

Try: $\langle -1, 1, -1 \rangle \times \langle 0, 1, 2 \rangle$ with either method

A. $\langle 3, 2, -1 \rangle$

$-\langle 0, -1, -2 \rangle \times \langle 1, 2, 0 \rangle$

A. $\langle 4, -2, 1 \rangle$

Geometry of cross products

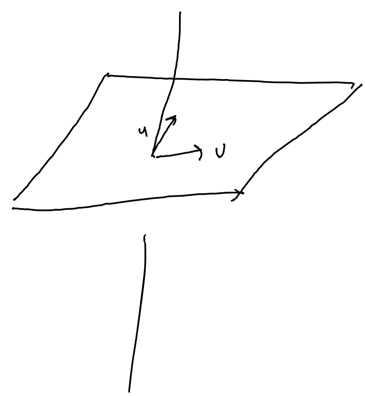
Facts. Let $\vec{u}$, $\vec{v}$ be vectors in $\mathbb{R}^3$

i. $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$

ii. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta)$

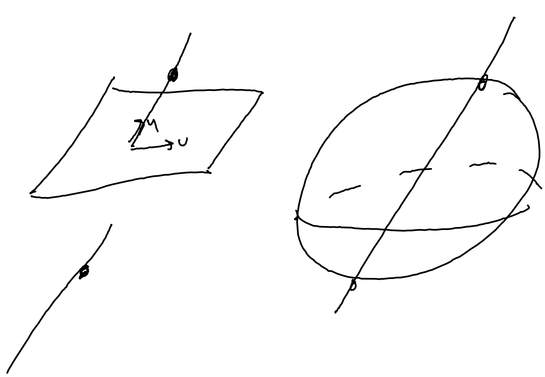
iii. $\{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$ is right handed

i.

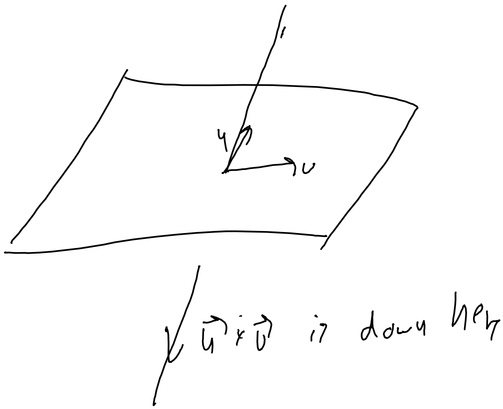


$\vec{u} \times \vec{v}$ is on this line

ii.



$\vec{u} \times \vec{v}$ is one of these 2 pts iii.



Scalar Triple product

Can we algebraically ($\to$ i.e. w/out drawing) check right handedness of 3 vectors $\{\vec{u}, \vec{v}, \vec{w}\}$?

Yes! This is precisely when $\vec{w}$ is the same direction as $\vec{u} \times \vec{v}$

when $\underbrace{(\vec{u} \times \vec{v}) \cdot \vec{w}}_{\text{"Scalar triple product of } \vec{u}, \vec{v}, \vec{w}\text{"}} > 0$

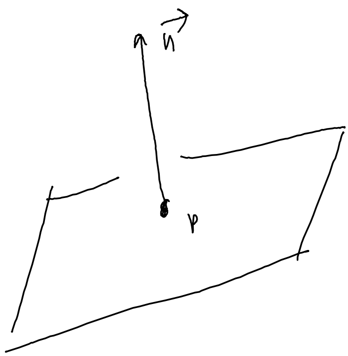
e.g. Is $\{\langle -1, 1, -1\rangle, \langle 0, 1, 2\rangle, \langle 0, 1, 3\rangle\}$ right handed?

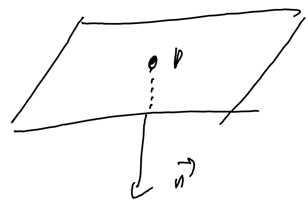
$\langle -1, 1, -1\rangle \times \langle 0, 1, 2\rangle = \langle 3, 2, -1\rangle$ from before, $\langle 3, 2, -1\rangle \cdot \langle 0, 1, 3\rangle = -1$, so no

Planes

A plane is a flat 2d object in $\mathbb{R}^3$.

This is described by a point p and a normal vector $\vec{n}$





The plane is the all the points q in $\mathbb{R}^3$ so that $\vec{pq}$ is orthogonal to $\vec{n}$.

Algebraically, this is equivalent to $\vec{pq} \cdot \vec{n} = 0$.

If $p = (p_1, p_2, p_3)$, $q = (x, y, z)$, $\vec{n} = \langle a, b, c \rangle$, this equation simplifies to $0 = \langle x - p_1, y - p_2, z - p_3 \rangle \cdot \langle a, b, c \rangle$

$$\boxed{0 = a(x - p_1) + b(y - p_2) + c(z - p_3)}$$

This can be simplified to $ax+by+cz=d$ if needed.

e.g. - $\vec{n} = \langle -1, 2, 1 \rangle$, $p = (4, 1, 5)$. what is the equation for the plane?

$$-1(x-4) + 2(y-1) + 1(z-5) = 0$$

$$-x + 4 + 2y - 2 + z - 5 = 0$$

$$-x + 2y + z = 3$$

- $\vec{n} = \langle 1, 0, 0 \rangle$, $p = (1, 2, 3)$

$(= \vec{i})$

$$x = 1$$

- If the eq'n for a plane is $3x + 7y + 15z = -20$,

what is a normal vector? $\langle 3, 7, 15 \rangle$

what is a point on the plane? $(0, 0, -\frac{20}{15})$.

Note the coefficients in front of x, y, z determine a normal vector.

- Find a plane through $(0,5,7)$ parallel to the xz plane.

xz plane is $y=0$, $\vec{n} = \langle 0,1,0 \rangle$,

so $y=5$ works

- Are $\underbrace{2x+3y+4z=1}$ and $4x+6y+8z=-1000$ parallel? ✓

How about ↓ and $-2x-3y-4z=0$? ✓

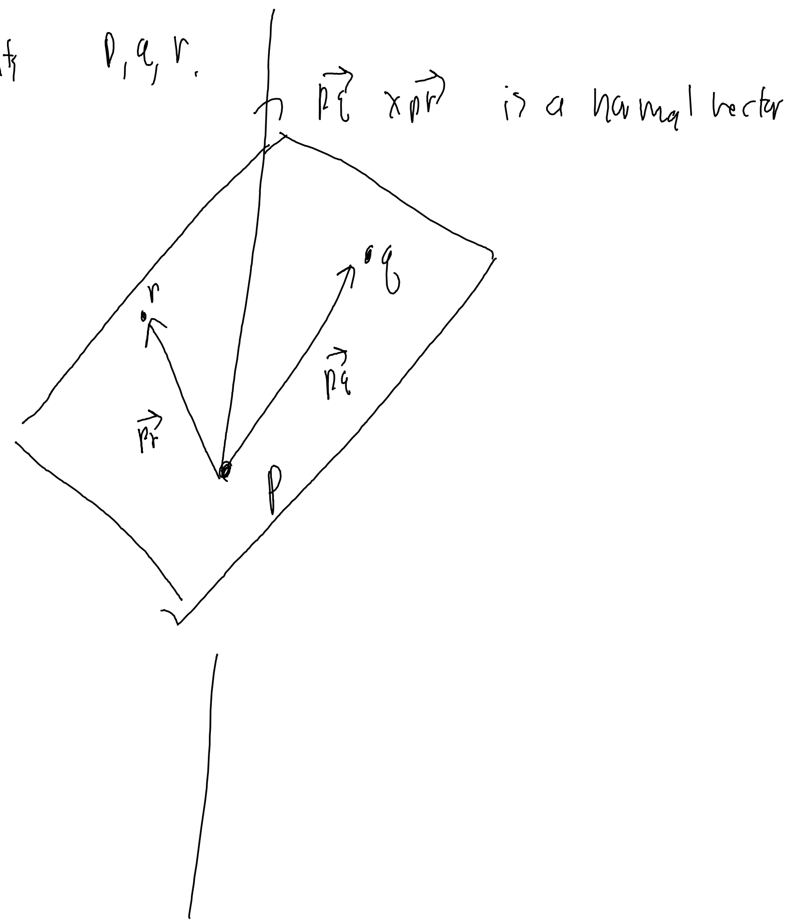
$2x-3y+4z=10$? ✗

Fact. Two planes are parallel precisely when they have parallel normal vectors

Recall that any two distinct points in $\mathbb{R}^2$ determine a unique line.

In $\mathbb{R}^3$, any three non-colinear points determine a unique plane.

Take points p, q, r.



e.g. $p = (1,0,0)$

$q = (2,2,3)$

$r = (0,0,5)$

$\vec{pq} = \langle 1,2,3 \rangle$

$\vec{pr} = \langle -1,0,5)$

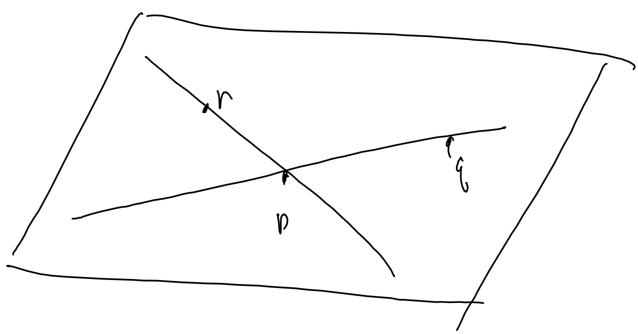
$\vec{pq} \times \vec{pr} = \langle 10, -8, 2 \rangle$, as before

to the plane is given by

$$10(x-1) - 8y + 2z = 0$$

$$10x - 8y + 2z = 10$$

Finally, if we have two distinct lines in $\mathbb{R}^3$ which intersect, there is a unique plane containing both lines.



Method. Find the intersection point. That's p

Find q, r, one on each line and distinct from p

Proceed as above, i.e. Compute $\vec{n} = \vec{pq} \times \vec{pr}$.