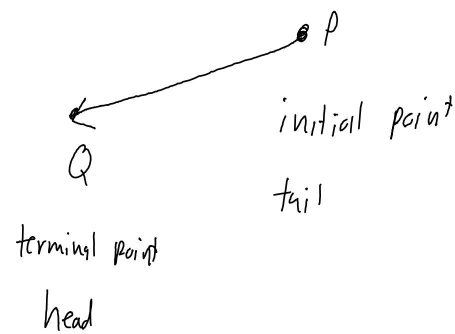
Vectors

Take two points P, Q in the plane $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$,
or in 3-space $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$

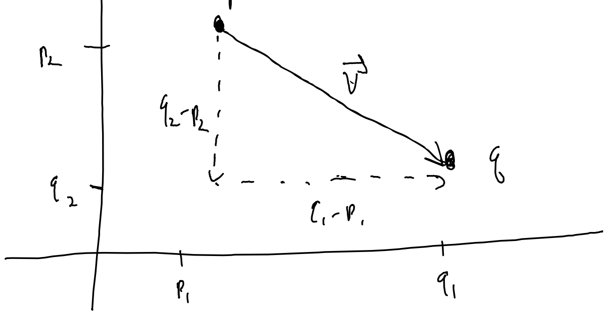
$\vec{v} = \overrightarrow{PQ}$ is the vector connecting them



Say $P = (p_1, p_2)$ $\quad Q = (q_1, q_2)$.

Then $\vec{v}$ has components $\langle q_1 - p_1, q_2 - p_2 \rangle$

$\left(P = (p_1, p_2, p_3),\ Q = (q_1, q_2, q_3) \Rightarrow \vec{v} = (p_1 - q_1, p_2 - q_2, p_3 - q_3) \right)$



e.g. i) $P = (1,1)$ $\quad Q = (5, 7)$.

What are the components of $\vec{v} = \vec{PQ}$?

$\vec{v} = \langle 5-1, 7-1 \rangle = \langle 4, 6 \rangle$

ii) Let $P = (0,-5,1)$, $\vec{v} = \langle 5, 1, 1 \rangle$

Say $\vec{v}$ is based at P. Where is its head?

$\langle q_1 - 0, q_2 - -5, q_3 - 1 \rangle = \langle 5, 1, 1 \rangle$

$\quad \|$

$\langle q_1, q_2 + 5, q_3 - 1 \rangle$

So $q_1 = 5$, $q_2 + 5 = 1$, $q_3 - 1 = 1$

$\therefore q_2 = -4$, $q_3 = 2$

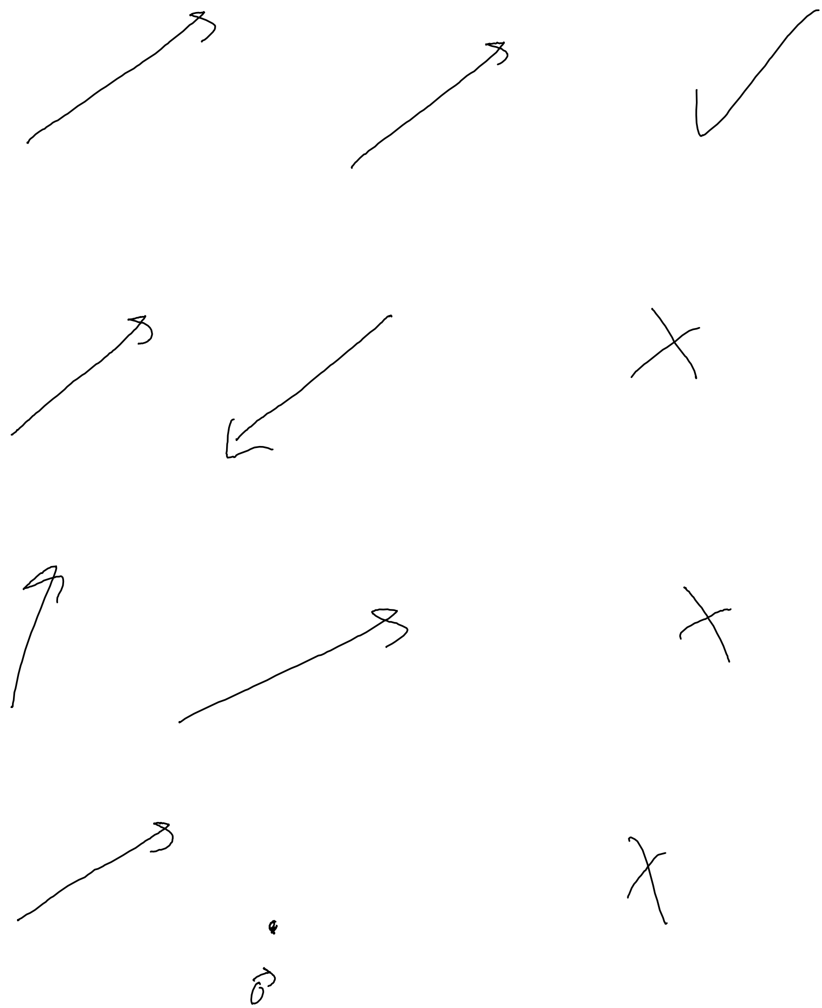
$Q = (5, -4, 2)$

iii) $Q = (1,1)$, $\vec{v} = \langle -1, -10 \rangle$. Say Q is the head of $\vec{v}$. Where is P?

$\langle -1, -10 \rangle = \langle 1 - p_1, 1 - p_2 \rangle$. $p_1 = 2$, $p_2 = 11$.

$\therefore P = (2, 11)$

Recall vectors $\vec{v}$ and $\vec{w}$ are <u>equivalent</u> if they're translations of each other



Let $\vec{v} = \langle 3, 1 \rangle$, what happens to the components

if I translate 2 units left,

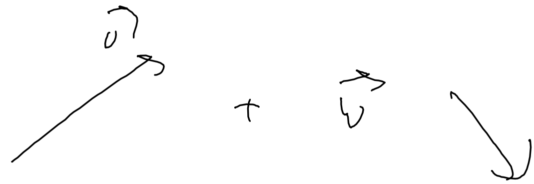
3 units up,

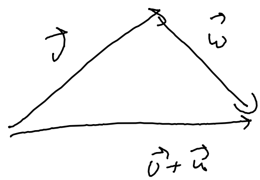
Both?

Nothing!

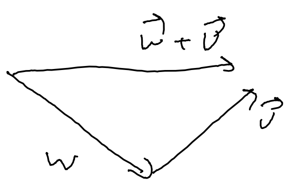
$\boxed{\text{Fact.}}$ $\vec{v}$ and $\vec{w}$ are equivalent precisely when they have the same components

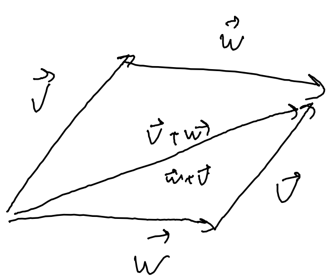
Addition:





On the other hand, $\vec{w}+\vec{v}$ is





So $\vec{v}+\vec{w} = \vec{w}+\vec{v}$

Algebraically, say $\vec{v} = \langle v_1, v_2 \rangle$, $\vec{w} = \langle w_1, w_2 \rangle$

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

$$= \langle w_1 + v_1, w_2 + v_2 \rangle$$

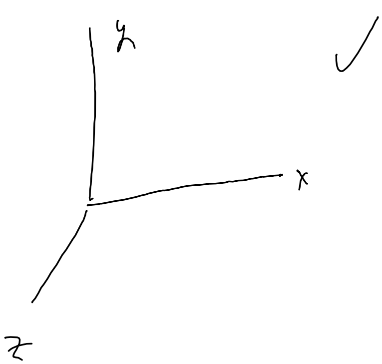
$$= \vec{w} + \vec{v}$$

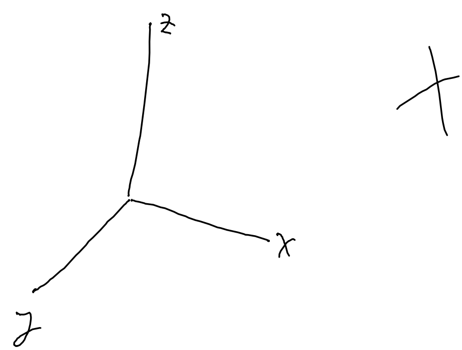
$$\langle 1, 2, 3 \rangle + \langle 4, 5, 6 \rangle = \langle 5, 7, 9 \rangle$$

$$\langle 4, 5, 6 \rangle + \langle 1, 2, 3 \rangle = \langle 5, 7, 9 \rangle.$$

3d

right hand rule





Also, recall $\vec{i} = \langle 1, 0, 0 \rangle$
$\vec{j} = \langle 0, 1, 0 \rangle$
$\vec{k} = \langle 0, 0, 1 \rangle$

$$\langle 3, 1, 2 \rangle = 3\vec{i} + 1\vec{j} + 2\vec{k}$$

Dot products

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{w} = \langle w_1, w_2, w_3 \rangle$

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$ a <u>scalar</u>

$\langle 1, 2, 3 \rangle \cdot \langle 4, 5, 6 \rangle = 4 + 10 + 18 = 32$

e.g.,

$\cdot$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

Key fact. $\vec{v} \cdot \vec{w} = \|v\| \|w\| \cos(\theta)$, θ the angle between $\vec{v}, \vec{w}$

(say, $0 \le \theta \le \pi$)

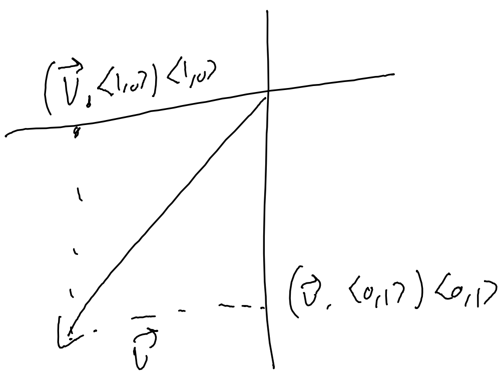
, , is $\vec{v} \cdot \vec{w}$ positive, negative, or 0?

(>0) (<0) (=0)

e.g. $\vec{i} \cdot \langle 1,2,3\rangle = 1$

$\vec{j} \cdot \langle 1,2,3\rangle = 2$

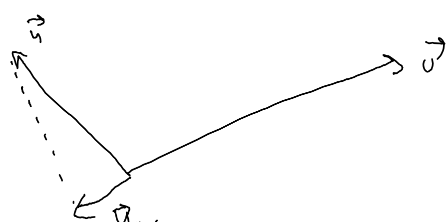
$\vec{k} \cdot \langle 1,2,3\rangle = 3$



In general,

let $\vec{v} \neq \vec{0}$, $\vec{u}$ a vector.

$$\underbrace{\vec{u}_{\parallel \vec{v}}}_{\text{parallel to } \vec{v}} = \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)}_{\text{scalar}} \underbrace{\vec{v}}_{\text{vector}}$$ is the projection of $\vec{u}$ along $\vec{v}$



e.g., $\vec{u} = \langle 4, -1, 0 \rangle$

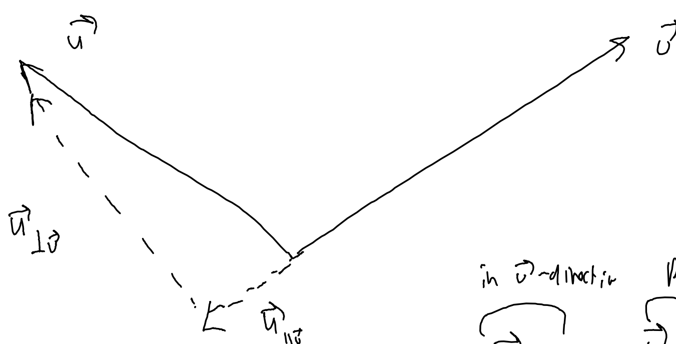
$\vec{v} = \langle 0, 1, 1 \rangle$

$$\vec{u}_{\parallel \vec{v}} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \cdot \vec{v}$$

$$= \frac{-1}{2} \langle 0, 1, 1 \rangle$$

$$= \langle 0, \tfrac{-1}{2}, \tfrac{-1}{2} \rangle$$

Back to our picture.



$$\vec{u} = \overbrace{\vec{u}_{\parallel \vec{v}}}^{\text{in } \vec{v}\text{-direction}} + \overbrace{\vec{u}_{\perp \vec{v}}}^{\text{perp to } \vec{v} \text{ direction}}$$

$\vec{u}_{\perp \vec{v}} = \vec{u} - \vec{u}_{\parallel \vec{v}}$, so

Indeed, $\vec{u}_{\perp \vec{v}} \cdot \vec{v} = (\vec{u} - \vec{u}_{\parallel \vec{v}}) \cdot \vec{v} = \vec{u} \cdot \vec{v} - \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \right) \cdot \vec{v} = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0$

Here, $\vec{u}_{\perp v} = \langle 4, -1, 0 \rangle - \langle 0, \frac{-1}{2}, \frac{-1}{2} \rangle$

$$= \langle 4, \frac{-1}{2}, \frac{1}{2} \rangle$$

$$\langle 4, 1, 0 \rangle = \langle 0, \frac{-1}{2}, \frac{-1}{2} \rangle + \langle 4, \frac{-1}{2}, \frac{1}{2} \rangle$$

$$\vec{u} = \vec{u}_{\parallel \vec{v}} \qquad \vec{u}_{\perp \vec{v}}$$

Cross products

Horrible formula:

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\vec{w} = \langle w_1, w_2, w_3 \rangle$$

Then $\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2)\,\vec{i} - (v_1 w_3 - v_3 v_1)\,\vec{j} + (v_2 w_2 - v_2 w_1)\,\vec{k}$

(If you know determinants this is $\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$)

Critical remarks

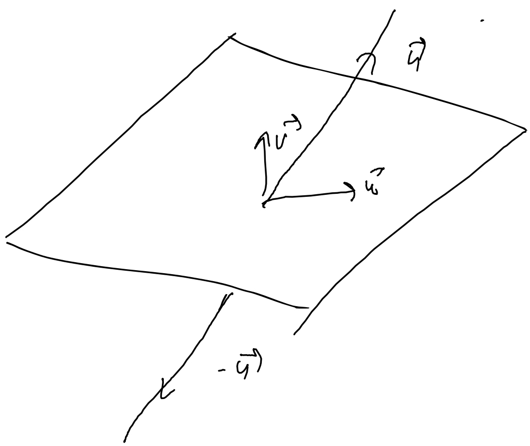
- $\vec{v} \times \vec{w}$ is a <u>vector</u> not a <u>scalar</u> like $\vec{v} \cdot \vec{w}$
- $\vec{v} \times \vec{w}$ <u>only</u> makes sense in $\mathbb{R}^3$, not in $\mathbb{R}^2$, $\mathbb{R}^4$, or anywhere else.

Purpose

- $\vec{v} \times \vec{w} \perp \vec{v}$ and $\vec{v} \times \vec{w} \perp \vec{w}$ line

- $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$ length

- $\{\vec{v}, \vec{w}, \vec{v} \times \vec{w}\}$ is right handed direction



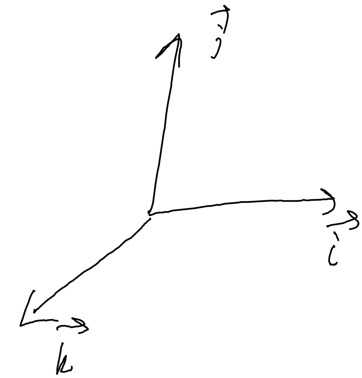
which of $\vec{y}$ / $-\vec{y}$ is $\vec{v} \times \vec{w}$?

A: $-\vec{y}$ by right handedness.

That formula is painful to memorize if you don't know determinants! way too many indices and signs.

My preference

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	0	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	0	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	0



Then just FOIL (aka distribute (aka use bilinearity)) across the cross product.

$$\langle 2, 3, 5\rangle \times \langle -1, 0, -3\rangle$$

$$= (2\vec{i} + 3\vec{j} + 5\vec{k}) \times (-1\vec{i} - 3\vec{k})$$

$$= (\underbrace{-2\,\vec{i}\times\vec{i}}_{0} - 6\,\vec{i}\times\vec{k}) + (-3\,\vec{j}\times\vec{i} - 9\,\vec{j}\times\vec{k}) + (-5\,\vec{k}\times\vec{i} - \underbrace{15\,\vec{k}\times\vec{k}}_{0})$$

$$= 6\vec{j} + 3\vec{k} - 9\vec{i} - 5\vec{j} = -9\vec{i} - 5\vec{j} + 9\vec{k}$$

$$= \langle -9, -5, 9\rangle.$$