Ch 13

§1. Plane vectors

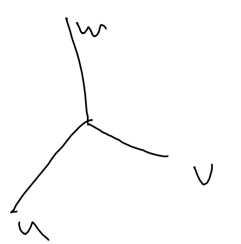
how do you $\vec{pq}$?

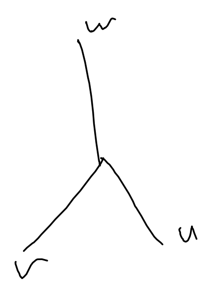
how do you add/subtract/scale vectors algebraically and geometrically?

what is $\|\vec{v}\|$?

§2. 3d

what is the right hand rule?





{u,v,w} R/L? {u,v,w} R/L?

{v,u,w} R/L? {v,u,w} R/L?

what is a parametrized curve? Can you parametrize a line? a circle?

$\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, $\vec{k} = \langle 0,0,1\rangle$

§3. Dot products

Critical formulas: i) $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

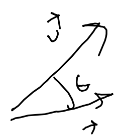
for $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$

If $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$

then $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$

ii) $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos \theta$

w/ $0 \le \theta \le \pi$ the angle between $\vec{v}$ and $\vec{w}$



$\vec{v} \cdot \vec{w}$ is

$> 0 \iff$ acute

$= 0 \iff$ orthogonal

$< 0 \iff$ obtuse

projection: $\vec{u}$ a vector, $\vec{v}$ a non zero vector

$\underbrace{\vec{u}_{\parallel \vec{v}}}_{\text{vector}} = \underbrace{\left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right)}_{\text{scalar}} \underbrace{\vec{v}}_{\text{vector}}$, parallel to $\vec{v}$

$\vec{u}_{\perp \vec{v}} = \vec{u} - \vec{u}_{\parallel \vec{v}}$, perpendicular to $\vec{v}$

$\vec{u} = \vec{u}_{\parallel \vec{v}} + \vec{u}_{\perp \vec{v}}$, the decomposition

§4. Cross product

<u>Warning</u>: only in 3d

output is a vector, not a scalar

$\vec{v} \times \vec{w} \neq \vec{w} \times \vec{v}$

$(\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ in fact$)$

to compute: $\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$ $\quad \vec{v} \times \vec{w}$

notice the order of $\vec{v}$ and $\vec{w}$!

for $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{w} = \langle w_1, w_2, w_3 \rangle$

Geometric facts:

i) $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$

ii) $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin(\theta)$

iii) $\{\vec{v}, \vec{w}, \vec{v} \times \vec{w}\}$ is a right handed system

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	0	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	0	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	0

Note. Consider $\{\vec{u}, \vec{v}, \vec{w}\}$

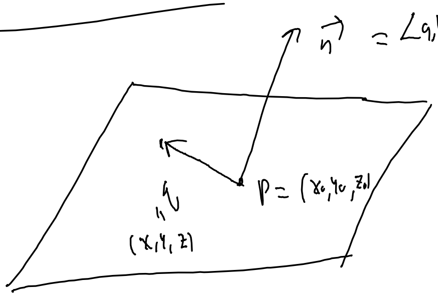
This is right handed if and only if

$\vec{u} \times \vec{v}$ and $\vec{w}$ are in the same direction,

i.e. $\underbrace{(\vec{u} \times \vec{v}) \cdot \vec{w}}_{\text{"Scalar triple product"}} > 0$

computed as $\det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$

§5 Planes



$\vec{n} = \langle a, b, c \rangle$

Q (x, y, z)

$P = (x_0, y_0, z_0)$

$\vec{PQ} \perp \vec{n}$

$\Updownarrow$

$\vec{PQ} \cdot \vec{n} = 0$

$\Updownarrow$

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$\Updownarrow$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\Updownarrow$

$ax + by + cz = ax_0 + by_0 + cz_0$

Given an equation for a plane, how can you find a normal vector?

how can you find a point on the plane?

On the other hand, given

i) point p, normal vector $\vec{n}$

ii) 3 points, not colinear

iii) 2 lines intersecting

find the equation for a plane

i) containing p w/ normal vector $\vec{n}$

ii) containing all 3 points

iii) containing both lines

Key: Basically always reduce to (i).

Ch. 12

§1 Parametric equations

What is a parametrization of a curve?

Know how to parametrize

- lines
- circles
- graphs of functions

Ch. 14

§1. vector valued functions

what is a vector valued function? in 2d? in 3d?

know how to parametrize some intersections of surfaces

methods: isolate a variable

find familiar parametrizations in some coordinates, like a circle

§2. Calculus of vector valued functions

How do you differentiate a vector valued function $\vec{r}$?

Rules: chain rule: $\frac{d}{dt}\vec{r}(g(t)) = \vec{r}\,'(g(t))\, g'(t)$

various product rules: $\frac{d}{dt}(f(t)\vec{r}(t)) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$

$$\frac{d}{dt}(\vec{r}_1(t) \cdot \vec{r}_2(t)) = \vec{r}_1\,'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2\,'(t)$$

$$\frac{d}{dt}(\vec{r}_1(t) \times \vec{r}_2(t)) = \underbrace{\vec{r}_1\,'(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{r}_2\,'(t)}_{\text{order is critical here!}}$$

How does the derivative relate to the tangent line? How can you parametrize the tangent line?

How do you integrate vector valued functions?

what is the vector valued version of the fundamental theorem of calculus?

§3. Arc length

Let $\vec{r}(t)$ be defined on $a \le t \le b$

The arclength of $\vec{r}$ from a to b is

$$\int_a^b \|\vec{r}\,'(t)\|\,dt$$

Let $s(t) = \int_a^t \|\vec{r}\,'(t)\|\,dt$, which inputs time and outputs distance traversed along the curve via $\vec{r}$ starting at $\vec{r}(a)$.

Then the speed of $\vec{r}(t)$ is $s'(t) = \|\vec{r}\,'(t)\|$ by FTC

Steps to arc length parametrize.

" Find an arc length parametrization for $\vec{r}(t)$ starting from p "

Step 0. Find the starting time, i.e. Solve $\vec{r}(t_0) = p$ for t_0

Step 1. Evaluate the arc length integral $s = g(t) = \int_{t_0}^t \|\vec{r}\,'(u)\|\,du$

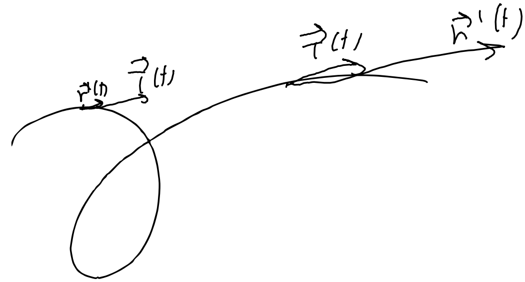
Step 2. Compute $t = g^{-1}(s)$, i.e. solve for t in the equation $s = g(t)$ from step 1.

Step 3. Compute $\vec{r}_1(s) = \vec{r}(g^{-1}(s))$, i.e. plug in $t = g^{-1}(s)$ for t in the given formula for $\vec{r}(t)$.

$\vec{r}_1(s)$ is the arc length parametrization

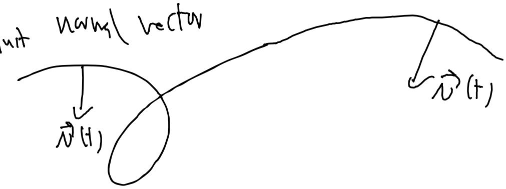
§4. Curvature

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ the unit tangent vector



the unit normal vector

$\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$



(note: $\vec{N}$ points in the direction $\vec{T}$ changes)

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$, the unit binormal vector

$\{\vec{T}(t), \vec{N}(t), \vec{B}(t)\}$ is the Frenet frame to $\vec{r}(t)$.

Curvature: Theoretical / conceptual definition

$$\kappa(s) = \left\| \frac{dT}{ds} \right\|$$

Computational definition

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

If $\vec{r}(t) = \langle x(t), y(t) \rangle$ in $\mathbb{R}^2$, then

curvature can be computed via

applied to $\langle x(t), y(t), 0 \rangle$

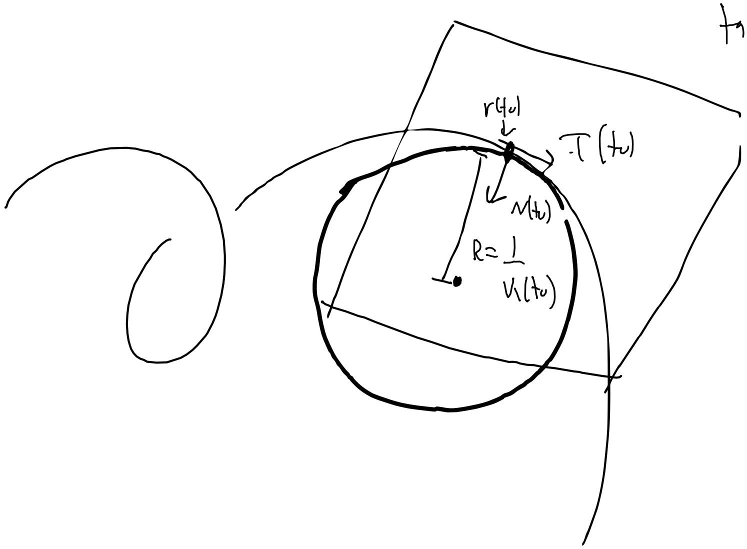
2 key examples

The curvature of a line is identically 0

the curvature of a circle of radius R is identically $\frac{1}{R}$

Osculating plane: At $t = t_0$, the osculating plane for $\vec{r}(t_0)$ is the plane through the point $\vec{r}(t_0)$ with the two vectors $\vec{T}(t_0)$, $\vec{N}(t_0)$, so a normal vector is $\vec{B}(t_0)$.

osculating circle: theoretical definition: the circle of largest radius in the osculating plane which is tangent to $\vec{r}(t_0)$.



formula: The radius of the osculating circle is $1/\kappa(t_0)$, for $\kappa(t)$ the curvature of $\vec{r}(t)$

The center is $\vec{r}(t_0) + \frac{1}{\kappa(t_0)} \vec{N}(t_0)$

§5. Motion in 3-Space

- Given $\vec{r}(t)$, what is its velocity?,
  its acceleration?,
  its speed?

- Given $\vec{a}(t)$, $\vec{v}(0)$, and $\vec{r}(0)$, how would you compute $\vec{r}(t)$?

- what is the acceleration of uniform circular motion?

- Know how to decompose the acceleration vector $\vec{a}(t)$ into tangent and normal components

$$\vec{a}(t) = \underbrace{a_{\vec{T}}(t)\,\vec{T}(t)}_{\substack{= \\ \vec{a}(t)_{\parallel \vec{T}(t)} \\ = \\ (\vec{a}(t)\cdot\vec{T}(t))\vec{T}(t)}} + a_{\vec{N}}(t)\,\vec{N}(t)$$

change in speed

Fact. $a_T(t) = v'(t)$, $v(t) = \|\vec{r}\,'(t)\|$ the speed

$a_N(t) = \kappa(t)\, v(t)^2$

change in direction