Solutions

1. use induction to prove

$$1+3+5+\ldots+(2n-1)=n^2$$

for $n \geq 1$

Scratch. Check @ beginning
check "update"

$$1+3+5+\ldots+(2n-1)+\underbrace{(2(n+1)-1)}_{(2n+1)}$$

LHS incr $n+1 \rightarrow$ adding $2n+1$

RHS $(n+1)^2 = n^2+\underbrace{2n+1}$

Pf. $P(n) =$ " $1+3+\ldots+(2n-1)=n^2$ "

Base case. $P(1)$. $1 = 1^2$, so $P(1)$ holds.

Ind step. Suppose $P(n)$ true for some $n \geq 1$.

WTS $P(n+1)$ true.

$(?)$

<u>wts</u> $1+3+\dots+(2n-1)+2(n+1)-1=(n+1)^2$

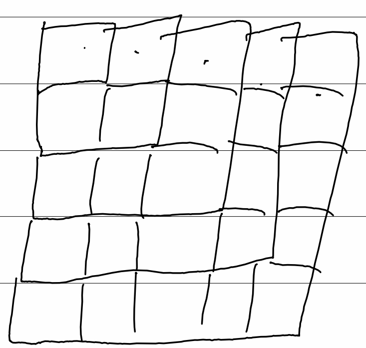
$1+3+\dots+(2n-1)+(2n+1)$

(ind hypothesis) $= n^2+2n+1$

$= (n+1)^2$

so $P(n+1)$ holds

Conclusion. By induction, $P(n)$ holds for all $n \geq 1$.



i- mult:

1'. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right) = \underline{n+1}$

$n \geq 1$

sketch.

$$\frac{\cancel{\left(1+\frac{1}{1}\right)}\cancel{\left(1+\frac{1}{2}\right)}\cdots\cancel{\left(1+\frac{1}{n}\right)}\left(1+\frac{1}{n+1}\right)}{\cancel{\left(1+\frac{1}{1}\right)}\cdots\cancel{\left(1+\frac{1}{n}\right)}}$$

$$\|$$

$$1+\frac{1}{n+1}$$

$$\frac{n+2}{n+1} = \frac{n+1+1}{n+1} = 1+\frac{1}{n+1}$$

$$\frac{LHS(n+1)}{LHS(n)} = \frac{RHS(n+1)}{RHS(n)}$$ "update" same way

Pf. $P(n) =$ "$\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$"

Base case. $1+\frac{1}{1} = 1+1$, so $P(1)$ holds

Inductive step. Say $P(n)$ holds some $n \geq 1$.

wts $P(n+1)$

wts $\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right) = n+2$

$\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$

so $\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right) = (n+1)\left(1+\frac{1}{n+1}\right)$

$1+\frac{1}{n+1} = \frac{n+2}{n+1}$

so $n+1\left(1+\frac{1}{n+1}\right) = n+2$

so $\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{n+1}\right) = n+2$

Thus, $P(n+1)$ holds

conclusion. By induction, $P(n)$ holds for all $n \geq 1$.

$a_n = b_n \quad \forall n$

$a_{n+1} - a_n = b_{n+1} - b_n \longrightarrow \Sigma = ?$

$\frac{a_{n+1}}{a_n} = \frac{b_{n+1}}{b_n} \longrightarrow \Pi = ?$

2. $A = \{a, b, c, d, e\}$

$B = \{a, w, b, x, y, d\}$

$C = \{c, e, w\}$

$D = \{a, d\}$ $\qquad \{a, a\} = \{a\}$

a) i). $\underline{A \cap B} = \{a, b, d\}$

ii) $\underline{B \cup C} = \{a, w, b, x, y, d, c, e\}$

iii) $\underline{B - A} = \{w, x, y\}$

iv) $C \times D = \{(c, a), (c, d), (e, a), (e, d), (w, a), (w, d)\}$

$$
\begin{array}{ccc}
 & a & d \\
c & (c, a) & (c, d) \\
e & (e, a) & (e, d) \\
w & (w, a) & (w, d)
\end{array}
$$

$\leftarrow |C| \cdot |D| = 3 \cdot 2 = 6$

$\{\emptyset, \{d\}\}$

$= P(D - \{a\})$

v) $P(D) = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$ $\leftarrow 2^{|D|}$ elts $= 2^2 = 4$

Q

$P(X)$ $x \in X$

$A \subseteq X$ s.t. $x \in A \rightarrow \{x\} \cup$ one of

$A \subseteq X$ s.t. $x \notin A$ + w/o

$=$

$P(X - \{x\})$

b) $f: \mathbb{Z} \longrightarrow \mathbb{Z}$

$f(n) = \lfloor \frac{n}{2} \rfloor$ ← floor fn i) rounding down

$\lfloor 0.5 \rfloor = 0$ $\lfloor 0.999 \rfloor = 0$

$\lfloor -0.1 \rfloor = -1$

i. Is f one to one?

for all n,m in $\mathbb{Z}$ $f(n) = f(m) \Rightarrow n = m$

Try $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{m}{2} \rfloor \Rightarrow n = m$

False find $n \neq m$ s.t. $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{m}{2} \rfloor$

$\lfloor \frac{1}{2} \rfloor = 0$ so $f(1) = f(0)$

$\lfloor \frac{0}{2} \rfloor = 0$ $\therefore$ not 1-1

iii. Is f onto?

given m, find n s.t.

$f(n) = m$

$\lfloor \frac{n}{2} \rfloor = m$

$n = 2m$

$\lfloor \frac{2m}{2} \rfloor = \lfloor m \rfloor = m$

Write down

Observe $f(2m) = \lfloor \frac{2m}{2} \rfloor = \lfloor m \rfloor = m$

Thus, for all m in $\mathbb{Z}$, $f(2m) = m$, so f is onto.

3. a) R relation

$$\{(1,1), (1,3), (2,2), (3,1)\}$$

on the set $X = \{1,2,3\}$

Is R an equivalence relation?
if so, describe its corresponding partition
If not, state precisely which properties fail
and justify?

Reflexive, No, as $(3,3)$ missing

symmetry, This is symmetric

Transitive. $(1,3) \in R$, $(3,1) \in R$
$(3,1) \in R$, $(1,3) \in R$ $\Rightarrow (3,3)$ if transitive
as $(3,3)$ missing but $(3,1), (1,3) \in R$, R is not
transitive.

Aside

$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

Partition $F = \{\{1,3\}, \{2\}\}$

6. $X = \{a, b, c\}$

$F = \{\{a\}, \{b, c\}\}$

- Describe eq. rel. R associated to F by giving a set of ordered pairs
- Determine $[b]$

- xRy if x, y are in the same elt of the partition of F

$\{(a,a), (b,b), (b,c), (c,c), (c,b)\}$

$[b] = \{b, c\}$.

4. a) select Pres

VP

Secr

among 8 people if no one can hold multiple offices

A. $P(8,3) = 8 \cdot 7 \cdot 6$

b) # bit strings w/ length at least 5 and at most 6

A. $l=5$ or $l=6$

$\frac{}{2}\ \frac{}{2}\ \frac{}{2}\ \frac{}{2}\ \frac{}{2}$

$2^5 + 2^6$

c) 8 bit string begin w/ 00 or end w/ 1111 or both

$X = \{$8 bit strings starting w/ 00$\}$ $|X \cup Y|$

$Y = \{$8 bit strings ending w/ 1111$\}$

$|X \cup Y| = |X| + |Y| - |X \cap Y|$ 7 - 8

- $|X|$: 00 _ _ _ _ _ _

2^6

- $|Y|$ _ _ _ _ 1111

2^4

- $|X \cap Y| =$ 00 $\underbrace{\text{\_ \_}}_{2^2}$ 1111

A. $|X \cup Y| = 2^6 + 2^4 - 2^2$

d) # ways to $\boxed{\text{rank}}$ the top 5 books from a collection of 9

(MB, RJ, MC, JL, OD)

A. $\underline{P(9,5)} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$

e) # 16 bit strings w/ exactly 4 1's

A. C(16, 4)

1 _ 1 1 1 _ _ 1 1 _ 1 1 1 _ _ _

{3, 1, 4, 5, 8, 9, 11, 12, 13}

5. a) strings formed by arrangements

(1 2 3 4 5 6 7) (1, 2, 4, 3, 5, 6, 7)
PEPPERS

3 P's   2 E's   1 R   1 S

By the multinomial coefficient formula, there are

$$\frac{7!}{3!\,2!\,1!\,1!}$$

$$\|$$

$$\underset{P's}{C(7,3)} \cdot \underset{E's}{C(4,2)} \cdot \underset{R's}{C(2,1)} \cdot \underset{S's}{C(1,1)}$$

b) # sols in $\mathbb{Z}^{\geq 0}$ to

$$x_1 + x_2 + x_3 + x_4 = 29$$

s.t. $x_1 > 0$, $x_2 > 2$, $x_3 \geq 3$, $x_4 \geq 0$

$\underline{x_1 \geq 1}$, $\underline{x_2 \geq 3}$, $\underline{x_3 \geq 4}$, $\underline{x_4 \geq 0}$

$1 + 3 + 4 + 0 = 8$

8 choices already made

Thus, there 21 objects left to pick

21 objects
3 dividers

24 position
3 dividers

$C(24,3)$

$x_1 \geq 1$ $\quad x_2 \geq 3$ $\quad x_3 \geq 4$ $\quad x_4 \geq 0$

$y_1 = x_1 - 1$ $\quad y_2 = x_2 - 3$ $\quad y_3 = x_3 - 4$ $\quad y_4 = x_4 - 0$

$x_1 + x_2 + x_3 + x_4 = 29$ given ineq

$\Updownarrow$

$y_1 + y_2 + y_3 + y_4 = 21$ $\quad y_i \geq 0$

$\downarrow$

$C(21+3, 3)$