- Binomial coefficients
- $\boxed{\text{Recurrence relations}}$
- Pigeonhole principle

Recurrence relations

a a 1 4

0 1 2 2

$$\begin{cases} a_0 = 5,\ a_1 = 17 \\ a_{n+2} = f(a_{n+1}, a_n) \end{cases}$$

$a_0 = 1$ $\longrightarrow$ $a_0 = 1$

$a_n = 2 \cdot a_{n-1}$. $a_1 = 2 \cdot a_0$

for $n \geq 1$ $= 2 \cdot 1$

$= 2$

$a_2 = 2 \cdot a_1$

$= 2 \cdot 2$

$= 4$

Hypothesis. $\underline{a_n = 2^n}$. for all $n \geq 0$

$a(n)$ a_n

$a_0 = 1$

$a_n = 2a_{n-1}$

Prop. $a_n = 2^n$ for $n \geq 0$

P.f. Base case. $a_0 = 1$ $\therefore a_0 = 2^0$

$2^0 = 1$

Ind. step. Suppose $a_n = 2^n$ some $n \geq 0$

wts $a_{n+1} = 2^{n+1}$

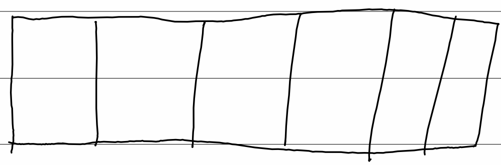
$a_{n+1} = 2 \cdot a_n$

By ind hyp. $2a_n = 2 \cdot 2^n = 2^{n+1}$

so indeed, $a_{n+1} = 2^{n-1}$

$\therefore$ by induction, $a_n = 2^n$ for all $n \geq 0$.

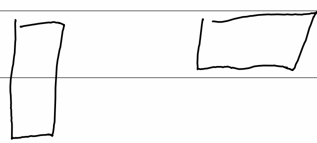
$2 \times n$ grid



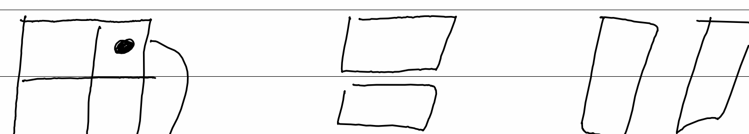
2×6

cover all squares without overlaps

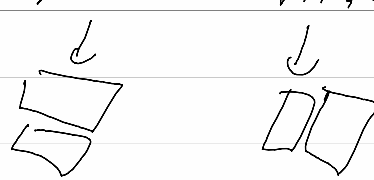
a_n = # ways to tile a $2 \times n$ grid w/ 2×1 dominoes



$n = 2$.



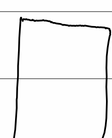
cov. by horiz or vertical domino



$a_2 = 2$.

$a_1 = ?$

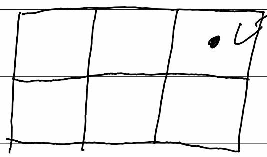
$a_1 = 1$

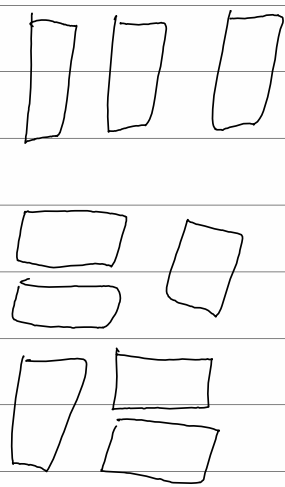


$a_1 = 1$ $a_2 = 2$

$a_3 = ?$

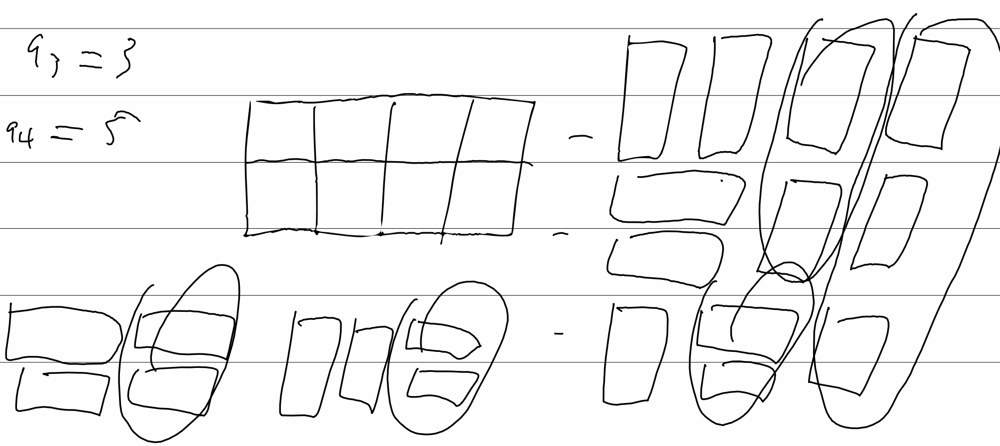
cov by vert domino or by horizontal





$a_3 = 3$

$a_4 = 5$



$a_n =$ # ways to tile $2 \times n$ grid

$a_1 = 1$ (OEIS)

$a_2 = 2$

$a_3 = 3$

$a_4 = 5$

vibes of fibonacci

$f_2 = 2, \quad f_1 = 1 \quad (f_0 = 1)$

$f_{n+2} = f_{n+1} + f_n$

$f_1 = 1, \quad f_2 = 2, \quad f_3 = 1 + 2 = 3, \quad f_4 = 3 + 2 = 5$

$f_5 = 5 + 3 = 8$

$f_6 = 8 + 5 = 13$

$\vdots$

Hypothesis. $a_n = f_n$ for all $n \geq 1$

Prop. $a_n = f_n$ for $n \geq 1$

Pf. By induction

Base case. $a_1 = 1$, $f_1 = 1$ $\quad a_1 = f_1$

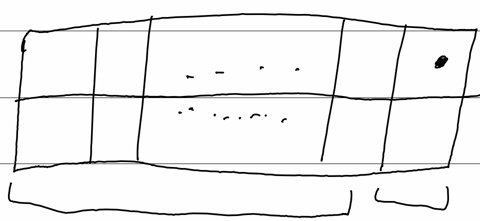
$a_2 = 2$, $f_2 = 2$ $\quad a_2 = f_2$

$$\left(\begin{array}{l} P(1) \\ P(1) \Rightarrow P(2) \end{array} \right)$$

Ind step. Suppose $a_n = f_n$ $\quad n \geq 1$

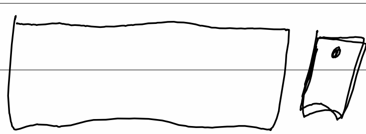
goal. $a_{n+1} = f_{n+1}$

$2 \times (n+1)$



$n \qquad 1$

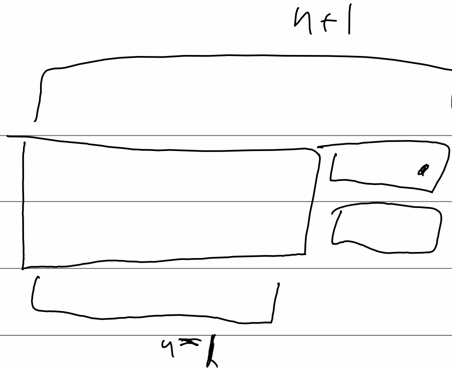
Case 1. • cov by vertical domino



a_n many ways to tile this

n

Case 2. • cov by horizontal domino



a_{n-1} ways to tile this

Case 1. a_n tilings

Case 2. a_{n-1} tilings

$$a_{n+1} = \underbrace{a_n}_{= f_n} + \underbrace{a_{n-1}}_{f_{n-1}}$$

$$a_{n+1} = f_n + f_{n-1} = f_{n+1}$$

Pigeonhole

17 pigeons 5 holes

some hole will have at least 2 pigeons

Question. Are there 2 people in LA (city not county)
w/ the exact same number of hairs
on their scalp?

What data do you need?

- pop LA
- upper bound # hairs on a person's scalp

What relation between these two variables would make the answer true?

pop LA > # hairs on scalp

census.gov - City of LA has $\approx$ 3.8 million people
Harvard bionumbers - at most 150000 hairs on a scalp

True!
There are at least 2 people in LA w/ the same # of hairs on their head by Pigeonhole principle

$$\text{people in LA} \longrightarrow \{0, \ldots, 150000\}$$
$$\text{person} \longmapsto \#\text{ hairs on their scalp}$$

Are there 3 people w/ same # of hairs?

"Third form" of pigeonhole
$f: X \longrightarrow Y$ $\quad |X| = n$, $|Y| = m$
Then there are $\lceil \frac{n}{m} \rceil$ in X mapping to the same elt in Y
$x_1, \ldots, x_{\lceil \frac{n}{m} \rceil}$ in X distinct
s.t. $f(x_1) = \ldots = f(x_{\lceil \frac{n}{m} \rceil})$

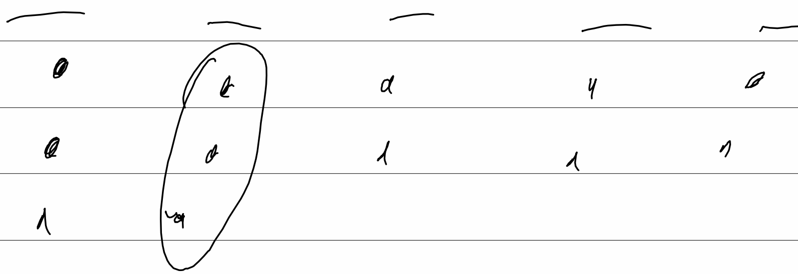
imagine conspiratorial pigeons

↓

maximally spread out

12 pigeons

5 holes



3 pigeons

$$3 = \left\lceil \frac{12}{5} \right\rceil = \lceil 2.4 \rceil$$

Capr. $$\left\lceil \frac{3.8 \text{ mill}}{150001} \right\rceil = 26$$

there are 26 people in LA w/ the same # of hairs on their head

Binomial coefficients

Thm. $(a+b)^n = \sum_{k=0}^{n} \underbrace{C(n,k)}\, \underline{a}^k b^{n-k}$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)(a+b)(a+b)(a+b) \cdots (a+b)$

$\underset{2^n \,\|}{(1+1)^n} = \sum_{k=0}^{n} C(n,k)$

$2^n = \underline{\sum_{k=0}^{n}} \underline{C(n,k)}$

$\downarrow \qquad\qquad \downarrow$

$2^n = |P(X)|$

$|X| = n$

Case k. Pick a k-sized subset of X

$C(n,k)$ many

$\#$ subsets of X $= \sum_{k=0}^{n} \#$ subsets of X of size k

$\|$

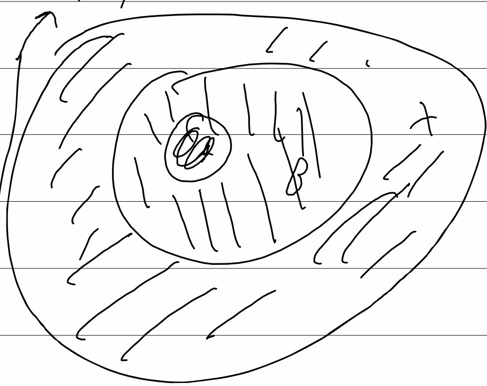
$2^n \qquad = \sum_{k=0}^{n} C(n,k)$

$3^n = (1+2)^n = \sum 2^{n-k} C(n,k)$

$|X| = n$

$\left| \{ (A, B) \in P(X)^2 \mid A \subseteq B \} \right| = 3^n$

(A, B) s.t. $A \subseteq B \subseteq X$



$(A, B-A, X-B)$

$= \left| \{ (A, B, C) \in P(X)^3 \mid \begin{array}{l} A \cup B \cup C = X \\ A \cap B = \phi \\ A \cap C = \phi \\ B \cap C = \phi \end{array} \} \right|$

$= 3^n$

$x \in X$, 3 possibilities of which set to be in per $x \in X$

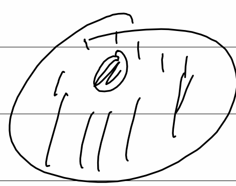
$$3^n = \sum_{k=0}^{n} 2^{n-k} C(n,k)$$

$$2^{n-k} C(n,k)$$

pick a subset A of size k in X

pick a subset C of X − A

$(A, A \cup C)$



pairs $(A,B) \mid A \subseteq B, |A| = k$

4^n ?

5^n ?

6^n ?

$\vdots$