OH Th 4PM

Midterm 1 Tues Jul 5

Integer partitions

eg. "partitions of 7"

Q. How many ways to solve $x_1 + x_2 + x_3 = 7$
when $x_1, x_2, x_3 \in \mathbb{Z}$, $x_i \geq 0$.

x_1 = apples
x_2 = bananas
x_3 = melons

? + ? + ? = 7

apples | bananas | melons

x_1 x_2 x_3

$$\overbrace{*}^{x_1} \;|\; \overbrace{*\;*\;*\;*}^{x_2} \;|\; \overbrace{*\;*}^{x_3}$$

$x_1 = 1$ $\quad$ $x_2 = 4$ $\quad$ $x_3 = 2$

$1!$ $\quad$ $4!$ $\quad$ $2!$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

$C(9,2)$

Star and bar diagram

Solution for $x_1+x_2+x_3=7$ in $\mathbb{Z}^{\geq 0}$

$C(9,2)$

$$x_1+x_2+x_3 = 7 \quad \text{in } \mathbb{Z}^{\geq 0}$$

7 $*$'s $\quad$ How many ways to arrange

2 $|$'s $\quad$ these?

Once we pick the location of $|$,
the $*$'s are forced

$|$ $|$ _ _ _ _ _ _ _

$C(9,2)$ ways to pick 2 elements from 9

Q. # sols to $\underline{x_1+x_2+x_3=7}$ s.t. $x_i \in \mathbb{Z}^{\geq 0}$

1. x_1 = # apples
   x_2 = # bananas
   x_3 = # melons

$$\underbrace{(1+1+1)}_{x_1}+\underbrace{(1+1)}_{x_2}+\underbrace{(1+1)}_{x_3}$$

$$* \quad * \;\Big|\; * \quad * \;\Big\|\; * \quad * \quad *$$

$x_1 = 2 \qquad x_2 = 2 \qquad x_3 = 3$

$$\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$\|\; \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$x_1 = 0$
$x_2 = 0$
$x_3 = 7$

$$\begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}$$

<u>Star and bar diagram</u> $\longleftrightarrow$ <u>sols $x_1+x_2+x_3=7$ in $\mathbb{Z}^{\geq 0}$</u>

Star + bar diagram   7 *'s

2 |'s

alphabet $\{*, |\}$

a, b

M, J

Martian, Jovian

$\{2, 9\}$

_ | _ _ _ _ _ _ |

1 2 3 4 5 6 7 8 9

count # ways to arrange 2 |'s in 9 buckets

$C(9,2)$ = # of 2 elt subsets of $\{1, \ldots, 9\}$

$$\frac{9 \cdot 8}{2} = 36$$

6. $x_1 + x_2 + x_3 = 7$, $x_i \in \mathbb{Z}^{\geq 0}$

How many sol's have

$x_1 \geq 0$ $\boxed{x_1 \geq 1}$

$x_2 \geq 1$ $\boxed{x_2 \geq 2}$

$x_3 \geq 2$ $\boxed{x_3 \geq 3}$

$x_1 + x_2 + x_3 = 7$

x_1 = apples x_2 = bananabananas

x_3 = melons

extra $\rightarrow y_1 = x_1 - 1$

extra $\rightarrow y_2 = x_2 - 2$

extra $\rightarrow y_3 = x_3 - 3$

1 = min'l # apple

2 = min'l # of bananas

3 = min'l # melons

$x_1 + x_2 + x_3 = 7$

$y_1 + y_2 + y_3 = x_1 + x_2 + x_3 - (1+2+3)$

$= 7 - 6$

$= 1$

solving $x_1 + x_2 + x_3 = 7$ s.t. $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$

$\Updownarrow$

$y_1 + y_2 + y_3 = 1$ s.t. $y_i \in \mathbb{Z}^{\geq 0}$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ x - sols, $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$

$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\begin{pmatrix} x_1 - 1 \\ x_2 - 2 \\ x_3 - 3 \end{pmatrix}$ y - sols $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Count # sols to $y_1 + y_2 + y_3 = \underline{1}$ s.t. $y_i \in \mathbb{Z}^{\geq 0}$

\# $*$'s 1

\# $|$'s 2

$C(3, 2) = 3$

3 sol's y - sols, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

x - sols, $\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

$x_1 + x_2 + x_3 = 100000$

$x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$

$y_1 + y_2 + y_3 = \underline{99994}$

99994 $*$'s

2 $|$'s

$C(99996, 2) \approx$ 100 million

Exc. Take variables x, y, z, w

Count number of monomials w/ leading coeff 1 of degree 72 in these variables.

$x^{70} y z \qquad x y z w^{69}$

(dim v.s. of deg 72 monomials)

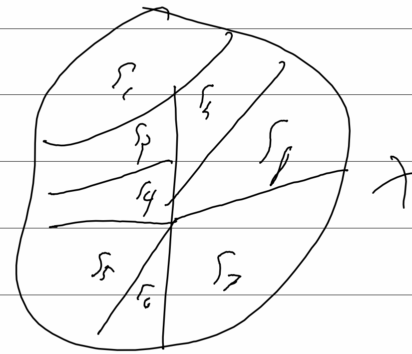
X set

$S_1, \dots, S_k \quad S_i \cap S_j = \emptyset \quad i \neq j$

$\bigcup_{i=1}^{n} S_i = X$

$\uparrow$

$\{\text{equivalence relation on } X\} \longleftrightarrow \{\text{partition on } X\}$



$R \longmapsto$ $x \in X$, we let

$[x] = \{y \text{ in } X \mid xRy\}$

claim: $\underline{[x]}$ is a partition of X

eq. class of x

$[x] = [y] \iff xRy$

$[1] = [2] \iff 1R2$

$X = \{1, 2, 3\}$

$\underline{R} = \{(1,1), (2,2), (3,3),$
$(1,2), (2,1)\}$

$[1] = \{1, 2\}$

$[2] = \{1, 2\}$

$[3] = \{3\}$

$X = \underbrace{\{1, 2\}} \cup \underbrace{\{3\}}$

$[0] = \{ n \in \mathbb{Z} \mid n R 0 \}$

$= \{ n \in \mathbb{Z} \mid n - 0 \text{ div } 3 \}$

$X = \mathbb{Z}$ $= \{ n \in \mathbb{Z} \mid n \text{ div } 3 \}$

$= \{ n \in \mathbb{Z} \mid \exists m \in \mathbb{Z} \text{ s.t. } 3m = n \} = \{ 3m \mid m \in \mathbb{Z} \}$

$x R y$ if $x - y$ is divisible by 3

Ex. This is an equiv

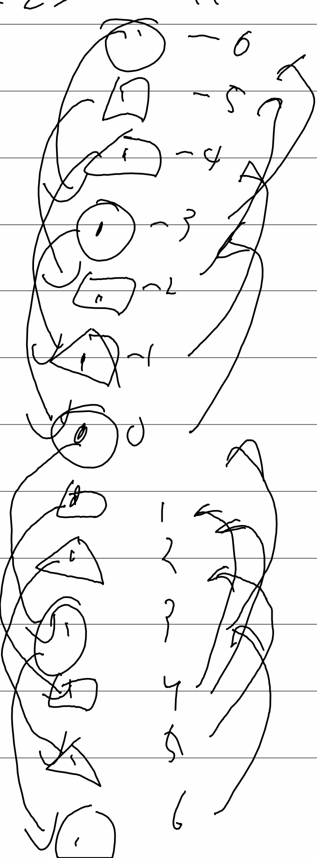
$[0] = \{ \text{multiples of } 3 \}$

$= \{ 3n \mid n \in \mathbb{Z} \}$

$[1] = \{ 1 + 3n \mid n \in \mathbb{Z} \} = \{ n \in \mathbb{Z} \mid n/3 \text{ has remainder } 1 \}$

$[2] = \{ 2 + 3n \mid n \in \mathbb{Z} \}$ 2

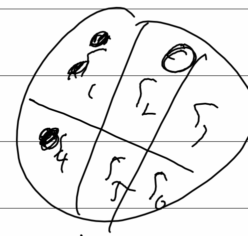
$\mathbb{Z} = [0] \cup [1] \cup [2]$



$R \longmapsto$ eq classes

Eq Rel $\longleftarrow\!\!\dashv$ Partition

$X = \bigcup_{i \in I} S_i$



$P = \{ S_i \mid i \in I \}$

$\bigcup P = X$

$R \longleftarrow\!\!\dashv P$

$x R y$ if there is $S_i \in P$ so that $x \in S_i$ and $y \in S_i$

eq classes of the P

$\mathbb{Z} = \{3n \mid n \in \mathbb{Z}\}$
$\cup \{3n+1 \mid n \in \mathbb{Z}\}$
$\cup \{3n+2 \mid n \in \mathbb{Z}\}$

<u>$x R y$</u> if $x, y \in \{3n \mid n \in \mathbb{Z}\}$
or $\in \{3n+1 \mid n \in \mathbb{Z}\}$
or $\in \{3n+2 \mid n \in \mathbb{Z}\}$

relation $R \longmapsto$ eq class

that ~~this~~ express $\longleftarrow\!\!\dashv P$

Eq Rel on $X \longleftrightarrow$ Part of X

$$\underline{\underline{R}} \;\longleftarrow\!\!\dashv\; P$$
$$\longmapsto P$$

$X = \{1, 2, 3, 4\}$

$\underline{\{1,2\}} = \underline{\{2,1\}}$

$\{1\} = \{1, 1, 1\}$

$(1, 2) \neq (2, 1)$

$\begin{pmatrix}1\\2\end{pmatrix} \neq \begin{pmatrix}2\\1\end{pmatrix}$

"distinct" $\longrightarrow$ permutations

$M_1 \quad M_2$

$M_2 \quad M_1$

$\underline{\text{ordered}} = \underline{\text{unordered}} \times \underline{\text{permutation}}$

Ex X set w/ 5 distinct elts
how many subsets of size 2 are in X?

$\{1,2\} = \{2,1\}$

how many ordered pairs of elts of X are there

$(1,2) \neq (2,1)$

7 pieces of fruit, apples, bananas, melons

apples ≥ banana ≥ melons

AAA BBB MMM

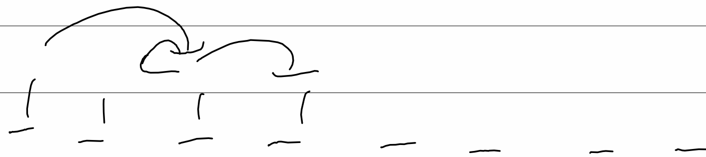
distinct

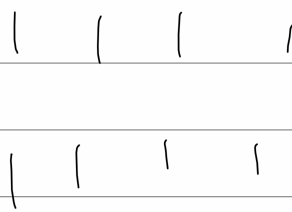
$A_1 A_2 A_3$ $B_1 B_2 B_3$ $M_1 M_2 M_3$

8 bits, how many 8 bit strings have 4 1's

11110000 = 11110000

8 bit strings w/ 4 1s





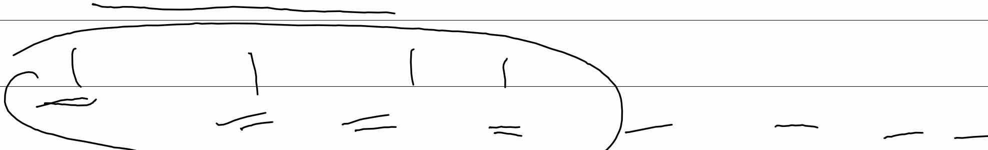
# of 4-elt subsets of an 8 elt set

C(8,4)

{1, 2, ..., 8}

{1, 2, 3, 4}

~~P(8,4)~~



(1, 4, 3, 2)

$1_1 \quad 1_4 \quad 1_3 \quad 1_2$ _ _ _ _

Ordered = unordered × permutation

MISSISSIPPI

1 M
4 I's
4 S's
2 P's

11 letters

$$\frac{11!}{1!\,4!\,4!\,2!}$$

8 bit strings w/ 4 1's $C(8,4)$

rearrangements 11110000

$$\frac{8!}{4!\,4!} = C(8,4)$$

11111000 $$\frac{8!}{5!\,3!} = C(8,5)$$

AP_2

— 1. AP_1P_2 $3!$

identical

$AP_1P_2 \leftrightarrow AP_2P_1$

$P_1AP_2 \leftrightarrow P_2AP_1$

$P_1P_2A \leftrightarrow P_2P_1A$

— 2. remove repetition divide by $2!$

$C(n,k)$ vs. repetition $\frac{n!}{n_1! \cdots n_\ell!}$

repetition formula. seq S w/ n elts

n_1 are identical of type 1

n_2 are identical of type 2 $n_1 + n_2 + \cdots + n_\ell = n$

$\vdots$

n_ℓ are identical of type ℓ

orderings of S $\frac{n!}{n_1! \cdots n_\ell!}$

$l = 2$

$C(n,k)$ = # k-elt subsets of an n elt set

$= \dfrac{n!}{k!\,(n-k)!}$

$X = \{1, 2, 3, 4, 5\}$ # 3-elt subsets of X

$C(5,3) = \dfrac{5!}{3!\,2!}$

subsets of X <—> assigning each elt of X a 0 or 1

$(X \longrightarrow \{0,1\})$

$S = \{1,2\}$

1 2 3 4 5

1 1 0 0 0

1 yes true

0 no, false

$S = \{2,1\}$ 1 1 0 0 0

subsets of size 3 <—> # 5bit strings w 3 1's

$\{1, 4, 5\}$ 10011

$\{1,2,3\}$ 11100

$X = \{a, b\}$ $\boxed{P(X) \longleftrightarrow \{\text{fns } \underline{X} \to \{0,1\}\}}$

$\emptyset$	00
$\{a\}$	10
$\{b\}$	01
$\{a,b\}$	11

$\{a,b,c\}$	$\begin{array}{ccc} a & b & c \\ \downarrow & \downarrow & \downarrow \end{array}$
ϕ	0 0 0
$\{a\}$	1 0 0
$\{b\}$	0 1 0
$\{c\}$	0 0 1
$\{b,a\}$	1 1 0
$\{a,c\}$	1 0 1
$\{b,c\}$	0 1 1
$\{a,b,c\}$	1 1 1

X has n elts

$$\left| \text{\# subsets of } X \text{ of size } k \right| = \left| \text{\# n-bit strings w/ } k \text{ 1's} \right|$$

$$\| \qquad\qquad\qquad \|$$

$C(n,k)$ $\qquad$ k 1's, $n-k$ 0's

$$\frac{n!}{k!\,(n-k)!}$$

$M_1 \quad M_2 \quad M_3 \quad M_4 \quad M_5$

J_1 — ——— — J_8