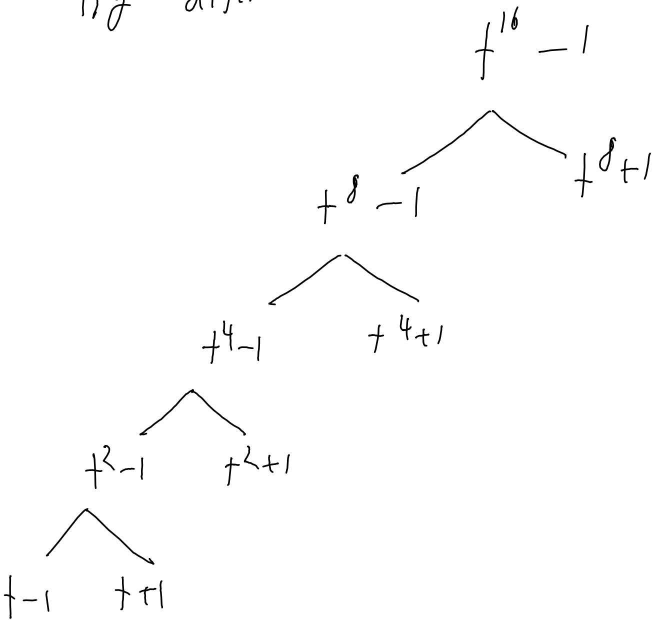
We consider $t^{16}-1$ over $\mathbb{Q}$.

By difference of squares we factor



t^2+1 irred / $\mathbb{Q}$ as it has no rat'l roots

t^4+1 irred/$\mathbb{Q}$ by HW

t^8+1?

$$(t+1)^8+1 = \sum_{i=0}^{8} \binom{8}{i} t^{n-i} + 1$$

The constant term is $(0+1)^8+1=2$.

The leading coefficient is $\binom{8}{0}=1$.

The intermediate coefficients are $\binom{8}{i}$ $1 \le i \le 7$ which are all even

(direct computation or find an involution)

So by Eisenstein's criterion, t^8+1 is irred. / $\mathbb{Q}$.

t^8+1 is the minimal polynomial of ζ_{16} over $\mathbb{Q}$,

ζ_{16} a primitive 16^{th} root of unity, e.g. $e^{2\pi i/16}$

$\mathbb{Q}(\zeta_{16})/\mathbb{Q}$ is the splitting field of $t^{16}-1$, $[\mathbb{Q}(\zeta_{16}):\mathbb{Q}]=8$.

$t-1$	Prim 1^{st} roots of unity
$t+1$	" 2^{nd} " " "
t^2+1	" 4^{th} " " "
t^4+1	" 8^{th} " " "
t^8+1	" 16^{th} " " "

Another factorization approach:

$$t^{16}-1 = \prod_{i=0}^{15}(t-\zeta_{16}^i)$$

There are 16 linear terms, so we can check all possible methods to recombine them until we get factorizations over $\mathbb{Q}$.

How many possibilities are there?

10,480,142,147

Considering symmetry makes this more systematic.

Idea, $\zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \dots, \zeta_{16}^{15}$ are all the primitive 16^{th} roots of unity and are all "algebraically indistinguishable"

(Formally, for $(k,16)=1$, $\zeta_{16} \longmapsto \zeta_{16}^k$ is an automorphism of $\mathbb{Q}(\zeta_{16})/\mathbb{Q}$)

$$(\mathbb{Z}/16\mathbb{Z})^\times \circlearrowright \{\zeta_{16}^i \mid 0 \le i \le 15\} = \mu_{16}$$

via $\bar{a} \cdot \zeta_{16}^i = \zeta_{16}^{ai}$

Conjugates under this action are, again, "alg indistinguishable" $(\mathrm{Aut}(\mu_{16}) \cong (\mathbb{Z}/16)^\times)$

Orbits of $(\mathbb{Z}/16\mathbb{Z})^\times$ on μ_{16} are

$t-1 \longleftrightarrow \{\zeta_{16}^{16}\}$ order 1

$t+1 \longleftrightarrow \{\zeta_{16}^{8}\}$ order 2

$t^2+1 \longleftrightarrow \{\zeta_{16}^4, \zeta_{16}^{12}\}$ order 4

$t^4+1 \longleftrightarrow \{\zeta_{16}^2, \zeta_{16}^{14}, \zeta_{16}^6, \zeta_{16}^{10}\}$ order 8

$t^8+1 \longleftrightarrow \{\zeta_{16}, \zeta_{16}^3, \dots, \zeta_{16}^{15}\}$ order 16

to factor this over $\mathbb{Q}(i)$.

Now suppose we want
we no longer want to consider $\underset{\zeta_{16}^4}{\overset{i}{\|}}$ and $\underset{\zeta_{16}^{12}}{\overset{-i}{\|}}$ in the same orbit

so consider $H = \{\bar{a} \in (\mathbb{Z}/16\mathbb{Z})^\times \mid \bar{a} \cdot i = i\}$
$= \{\bar{a} \in (\mathbb{Z}/16\mathbb{Z})^\times \mid 4a \equiv 4 \bmod 16\}$
$= \{\bar{a} \in (\mathbb{Z}/16\mathbb{Z})^\times \mid a \equiv 1 \bmod 4\}$

$$\left(= \ker\left(\begin{array}{ccc} (\mathbb{Z}/16\mathbb{Z})^\times & \longrightarrow & (\mathbb{Z}/4\mathbb{Z})^\times \\ \| & & \| \\ \mathrm{Aut}(\mu_{16}) & & \mathrm{Aut}(\mu_4) \\ \| & & \| \\ \mathrm{Aut}(\mathbb{Q}(\zeta_{16})) & \xrightarrow{\text{res}} & \mathrm{Aut}(\mathbb{Q}(i)) \end{array}\right)\right)$$

What are the H orbits of μ_{16}?

$t-1$	$\{\varphi_{16}^{10}\}$	1
$t+1$	$\{\varphi_{16}^{8}\}$	2
$t-i,\ t+i$	$\{\varphi_{16}^{4}\},\ \{\varphi_{16}^{12}\}$	4
$t^2-i,\ t^2+i$	$\{\varphi_{16}^{2}, \varphi_{16}^{10}\},\ \{\varphi_{16}^{6}, \varphi_{16}^{14}\}$	8
$t^4-i,\ t^4+i$	$\{\varphi_{16}, \varphi_{16}^{5}, \varphi_{16}^{9}, \varphi_{16}^{13}\},\ \{\varphi_{16}^{3}, \varphi_{16}^{7}, \varphi_{16}^{11}, \varphi_{16}^{15}\}$	16

$$\begin{tikzcd}
\mathbb{Q}(\varphi_{16}) \arrow[d, dash] \arrow[dd, dash, bend right=60, "H"'] \arrow[ddd, dash, bend left=60, "(\mathbb{Z}/16\mathbb{Z})^\times"] \\
\mathbb{Q}(\varphi_8) \arrow[d, dash] \\
\mathbb{Q}(\varphi_4) \arrow[d, dash, "(\mathbb{Z}/4\mathbb{Z})^\times"] \\
\mathbb{Q}
\end{tikzcd}$$