Transcendental Extensions

Let K/F and $\alpha \in K$. Recall that α is <u>transcendental</u> over F if for $f \in F[t]$ s.t. $f(\alpha) = 0$, $f = 0$.

Equivalently, $F[t] \longrightarrow K$ is injective.
$t \longmapsto \alpha$

Thus,

$$\begin{tikzcd} F(t) \arrow[r, hook] \arrow[dr, "\sim"'] & K \\ & F(\alpha) \arrow[u, hook] \end{tikzcd}$$

More generally,

Def. K/F and $S = \{s_i\}_{i \in I} \subseteq K$. S is algebraically independent over F if for $f \in F[\{t_i\}_{i \in I}]$ s.t. $f(\{s_i\}_{i \in I}) = 0$, then $f = 0$.

Equivalently, $F[\{t_i\}_{i \in I}] \longrightarrow K$ is injective, so
$t_i \longmapsto s_i$

$$\begin{tikzcd} F(\{t_i\}_{i \in I}) \arrow[r, hook] \arrow[dr, "\sim"'] & K \\ & F(S) \arrow[u, hook] \end{tikzcd}$$

we say that $F(S)/F$ is purely transcendental.

e.g. In $F(t_1, \ldots, t_n)/F$, $\{t_1, \ldots, t_n\}$ is algebraically independent$/F$.

e.g. in $F(t)$, t^2, t^3 are alg. dependent as $f(x,y) = x^3 - y^2$ has $f(t^2, t^3) = 0$.

Def. A max'l alg. indep$/F$ subset of K is called a transcendence basis of K/F.

Rmk. Let S alg. ind.$/F$. Then S is a transcendence basis $/F$ iff $K/F(S)$ is algebraic

$$\overbrace{K/F(S)}^{\text{algebraic}}/F$$
$$F(S)/F \text{ purely transcendental}$$

e.g. $\{t_1, \ldots, t_n\}$ a transcendence basis for $F(t_1, \ldots, t_n)/F$.

Thm. Let K/F, $T \subseteq K$ s.t. $K/F(T)$ algebraic and $S \subseteq T$ alg indep$/F$. Then $\exists$ $S \subseteq B \subseteq T$ s.t. B is a transcendence basis of K/F.

Pf. Consider $P = \{A \subseteq T \text{ alg. indep}/F\}$. Then $S \in P$, so $P \neq \emptyset$. Let $\mathcal{E} \subseteq P$ be a chain. Let $A = \bigcup_{C \in \mathcal{E}} C$. Then as polynomials only have finitely many non zero coefficients, A is alg. indep. $/F$. Hence, by Zorn's lemma, there is some $B \in P$ maximal.

We claim that B is a transcendence basis of K/F, so we wts that $K/F(B)$ is algebraic.

$$\underbrace{K/F(T)}_{\text{alg.}}/F(B)$$

Let $t \in T$. Then by maximality of B, t is algebraic over $F(B)$. Hence, $F(T)/F(B)$ is algebraic so $K/F(B)$ is algebraic $\square$

Cor. Take $S=\varnothing$, $T=K$. Then K/F admits a transcendence basis

Cor. Let K/F be finitely generated, so $K=F(T)$ for a finite set T. Then K/F admits a finite transcendence basis.

Def. The transcendence degree of K/F is the cardinality of a transcendence basis. This is denoted as $\mathrm{tr.deg}(K/F)$ or $\mathrm{tr.deg}_F(K)$.

We know transcendence bases exist, but we must still show well definition.

Thm. Any two transcendence bases of K/F have the same cardinality.

Pf. Let B, B' be transcendence bases of K/F.

Suppose WLOG $|B'| \leq |B|$.

Case 1. $|B|$ finite.

Let $B = \{\alpha_1, \ldots, \alpha_n\}$, $B' = \{\beta_1, \ldots, \beta_m\}$.

We induct on m.

$m=0$. Then $B' = \emptyset$ and $K/F(\emptyset) = F$ is algebraic, so $B = \emptyset$.

Let $m > 0$. By maximality, $B \cup \{\beta_1\}$ is alg. dep.,

so $\exists\ f \in F[s, t_1, \ldots, t_n]$ nonzero

s.t. $f(\beta_1, \alpha_1, \ldots, \alpha_n) = 0$.

As B is alg. indep. $/K$, f must contain an s term. Similarly, as β_1 is not algebraic over F, $f \notin F[s]$. Suppose then that f contains t_1. Thus, α_1 is alg. $/F(\beta_1, \alpha_2, \ldots, \alpha_n)$.

We claim $\{\beta_1, \alpha_2, \ldots, \alpha_n\}$ is a transcendence basis for K/F.

$K/F(\beta_1, \alpha_1, \ldots, \alpha_n)/F(\beta_1, \alpha_2, \ldots, \alpha_n)$

So we are left to show that

$\{\beta_1, \alpha_2, \dots, \alpha_n\}$ is alg. indep.

If not, let $g(\beta_1, \alpha_2, \dots, \alpha_n) = 0$,

$g \in F[t, s_2, \dots, s_n]$ nonzero.

As before, g contains a t term as

$\{\alpha_2, \dots, \alpha_n\}$ is alg indep $/F$.

But then β_1 is algebraic over $F(\alpha_2, \dots, \alpha_n)$

and α_1 is algebraic over $F(\beta_1, \alpha_2, \dots, \alpha_n)$

so α_1 is alg. $/F(\alpha_2, \dots, \alpha_n)$, contradicting

alg. indep. of B $/F$.

Thus, so $\{\alpha_2, \dots, \alpha_n\}$ and $\{\beta_2, \dots, \beta_m\}$ are

transcendence bases of $K/F(\beta_1)$. By

induction, $n = m$.

Case 2. $|B|$ infinite.

Let $\alpha \in B'$. Then α is alg. over $F(B)$, so

$\exists$ a finite subset $B_\alpha \subseteq B$ so that α is alg. $/F(B_\alpha)$.

Let $B^* = \bigcup_{\alpha \in B'} B_\alpha$. We claim $B^* = B$.

"$\subseteq$" clear

"$\supseteq$" Let $\beta \in B$. Then β alg $/F(B')$ and $F(B')/F(B^*)$ is algebraic. So β is algebraic over $F(B^*)$. Hence, B^* is alg indep $/F$ and $K/F(B^*)$ is algebraic, so B^* is a transcendence basis of K/F. Then by maximality of B^*, $B \subseteq B^*$ as desired.

Hence, $|B| = |B^*| \leq \sum_{\alpha \in B'} |B_\alpha| = |B'|$ $\square$

Prop. Let $K/E/F$. Then $\operatorname{tr\,deg}(K/F) = \operatorname{tr\,deg}(K/E) + \operatorname{tr\,deg}(E/F)$

Pf. Let B be a transcendence basis for K/E and C a transcendence basis for E/F.

$$
\begin{tikzcd}
& K \arrow[d, dash, "alg"] \\
& E(B) \arrow[dd, dash] \\
F(C \cup B) \arrow[ur, dash, "alg"] & \\
& E \\
F(C) \arrow[uu, dash] \arrow[ur, dash, "alg"'] &
\end{tikzcd}
$$

So take $B \cup C$ $\square$