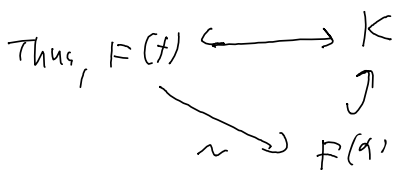


Transcendental Extensions

Let K/F and $\alpha \in K$, recall that α is transcendental over F if for $f \in F[t]$ s.t. $f(\alpha) = 0$, $f = 0$.

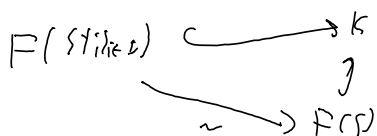
Equivalently, $F[t] \xrightarrow{\quad} K$ is injective,
 $t \mapsto \alpha$



More generally,

Def. K/F and $S = \{s_i\}_{i \in I} \subseteq K$, S is algebraically independent over F if for $f \in F[s_i\}_{i \in I}]$ s.t. $f(s_i\}_{i \in I}) = 0$, then $f = 0$.

Equivalently, $F[s_i\}_{i \in I}] \xrightarrow{\quad} K$ is injective, s.t.
 $s_i \mapsto s_i$



we say that $F(S)/F$ is purely transcendental,

e.g., In $F(t_1, \dots, t_n)/F$, $\{t_i \rightarrow t_n\}$ is algebraically

independent $/F$,

e.g., in $F(t)$, t^2, t^3 are alg. dependent as $f(x,y) = x^3 - y^2$ has $f(t^2, t^3) = 0$.

Def. A max'l alg. indep subset of K is called
a transcendence basis of K/F ,

Prop. Let S alg. ind. $/F$, then S is a transcendence basis $/F$ iff

$K/F(S)$ is algebraic

$$\underbrace{K/F(S)}_{\text{algebraic}} / F$$

e.g., $\{t_1, \dots, t_n\}$ purely transcendental
a transcendence basis for $F(t_1, \dots, t_n)/F$,

Thm. Let K/F , $T \subseteq K$ s.t. $K/F(T)$ algebraic
and $S \subseteq T$ alg. indep $/F$. Then $\exists S \subseteq B \subseteq T$ s.t.

B is a transcendence basis of K/F ,

Pf. Consider $\rho = \{A \subseteq T \text{ alg. indep. } /F\}$. Then $S \in \rho$, so $\rho \neq \emptyset$.

Let $\mathcal{C} \subseteq \rho$ be a chain. Let $A = \bigcup_{C \in \mathcal{C}} C$. Then as polynomials
only have finitely many non zero coefficients, A is alg. indep. $/F$.

Hence, by Zorn's Lemma, there is some $B \in \rho$ maximal.

we claim that B is a transcendence basis of K/F ,
 so we wts that $K/F(B)$ is algebraic.

$$\underbrace{K/F(T)}_{\text{alg.}}/F(B)$$

Let $t \in T$. Then by maximality of B , t is algebraic over $F(B)$.

Hence, $F(T)/F(B)$ is algebraic so $K/F(B)$ is algebraic $\square$

Cor. Take $S = \emptyset$, $T = K$. Then K/F admits a

transcendence basis

Cor. Let K/F be finitely generated, so $K = F(T)$ for a
 finite set T . Then K/F admits a finite transcendence
 basis.

Def. The transcendence degree of K/F is the
 cardinality of a transcendence basis, this is
 denoted as $\text{tr.deg}(K/F)$ or $\text{tr.deg}_p(K)$.

We know transcendence bases exist, but we must still
 show well definition.

Thm. Any two transcendence bases of K/F have the same cardinality.

Pf. Let B, B' be transcendence bases of K/F .

Suppose wlog $|B'| \leq |B|$.

Case 1. $|B|$ finite,

Let $B = \{\alpha_1, \dots, \alpha_n\}$, $B' = \{\beta_1, \dots, \beta_m\}$.

We induct on m .

$m=0$. Then $B' = \emptyset$ and $K/F(\emptyset) = F$ is algebraic, so $B = \emptyset$.

Let $m \geq 1$. By maximality, $B \cup \{\beta_1\}$ is alg. dep.,

so $\exists f \in F[s, t_1, \dots, t_n]$ nonzero

s.t. $f(\beta_1, \alpha_1, \dots, \alpha_n) = 0$.

As B is alg. indep. / K , f must contain an s term. Similarly, as β_1 is not algebraic

over F , $f \notin F[s]$. Suppose then that f contains t_1 . Thus, α_1 is alg. / $F(\beta_1, \alpha_2, \dots, \alpha_n)$.

We claim $\{\beta_1, \alpha_2, \dots, \alpha_n\}$ is a transcendence basis for K/F .

$K/F(\beta_1, \alpha_1, \dots, \alpha_n)/F(\beta_1, \alpha_2, \dots, \alpha_n)$

So we are left to show that

$\{x_1, x_2, \dots, x_n\}$ is alg. indep.

If not, let $g(x_1, x_2, \dots, x_n) = 0$,

$g \in F[t_1, t_2, \dots, t_n]$ non zero.

As before, g contains a t term as

$\{x_2, \dots, x_n\}$ is alg indep / F .

But then x_1 is algebraic over $F(x_2, \dots, x_n)$

and x_1 is algebraic over $F(x_1, x_2, \dots, x_n)$

so x_1 is alg. / $F(x_2, \dots, x_n)$, contradicting

alg. indep. of B / F .

Thus, $\{x_2, \dots, x_n\}$ and $\{x_2, \dots, x_m\}$ are

transcendence bases of $K/F(x_1)$. By

induction, $n=m$.

Case 2, $|B|$ infinite.

Let $\alpha \in B'$. Then α is alg. over $F(B)$, so

$\exists$ a finite subset $B_\alpha \subseteq B$ so that α is alg. / $F(B_\alpha)$.

Let $B^\star = \bigcup_{\alpha \in B'} B_\alpha$. We claim $B^\star = B$.

" $\subseteq$ " Clear

" $\supseteq$ " Let $\beta \in B$. Then β alg. / $F(B')$ and $F(B')/F(B^\star)$ is

algebraic. So β is algebraic over $F(B^\star)$. Hence, $B^\star$

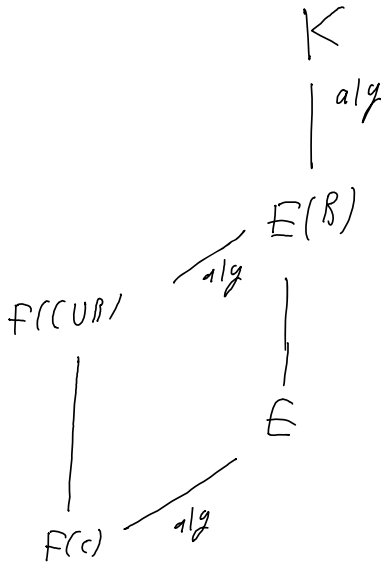
is alg indep / F and $|B|/|F(B^\star)|$ is algebraic, so $B^\star$ is a transcendence

basis of K/F . Then by maximality of $B^\star$, $B \subseteq B^\star$ as desired.

$$\text{Hence, } |B| = |B^*| \leq \sum_{\alpha \in B'} |B_\alpha| = |B'| \quad \square$$

Prop. Let $K/E/F$, Then $\text{trdeg}(K/F) = \text{trdeg}(K/E) + \text{trdeg}(E/F)$

Pr. Let B be a transcendence basis for K/E and C a transcendence basis for E/F .



So $F \not\supseteq K$ $B \cup C$

$\square$