33A
Discussion Worksheet 2

1. *Proof.* The first step is to translate this problem into a system of linear equations, which we can solve using row reduction (aka Gauss – Jordan elimination). Indeed, saying that $\vec{w}$ is a linear combination of $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ by definition means that there are some scalars x_1, x_2, x_3 so that $\vec{w} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$. If one is comfortable with the algebra of matrix – vector multiplication, we can immediately interpret this vector equation as $A\vec{x} = \vec{w}$ where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A = \begin{pmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{pmatrix}$$

Indeed, $A\vec{x}$ is then equal to $x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$. If one is not quite comfortable with this, we can also work directly. Let's take our equation $\vec{w} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$ and expand it out with our given coordinates for these vectors.

$$\begin{aligned} \vec{w} &= x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} \\ \begin{pmatrix} -5 \\ 11 \\ -7 \end{pmatrix} &= x_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ -2x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2x_3 \\ 0 \\ 8x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + 2x_3 \\ -2x_1 + x_2 \\ 2x_1 + x_2 + 8x_3 \end{pmatrix} \end{aligned}$$

which corresponds to solving the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 1 & 0 & 11 \\ 2 & 1 & 8 & -7 \end{array}\right)$$

Now, one has to row reduce this matrix. I won't write all the details for this, since it's a bit tedous to type and not conceptually new. Once one row reduces this, we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The last equation corresponds to $0x_1 + 0x_2 + 0x_3 = 1$, which is impossible as $0 \neq 1$. That is, this is an inconsistent system. So what does this have to do with the original problem? The inconsistency of this system means that we *cannot* find a solution

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

to the system of linear equations. But solving that system of linear equations is exactly the same as finding scalars x_1, x_2, x_3 so that $x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} = \vec{w}$. That's why we constructed the system as we did! So we conclude that no such scalars x_1, x_2, x_3 exist, so $\vec{w}$ is *not* a linear combination of $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$. □

2. *Proof.*
   - The notation "$w \in S$" means "w is an element of the set S". S is by definition the span of $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$. As such, its elements are exactly vectors of the form $x_1\vec{v_1}+x_2\vec{v_2}+x_3\vec{v_3}$ for some scalars x_1, x_2, x_3. So to ask if $w \in S$ is the same as asking if we can find scalars x_1, x_2, x_3 so that $\vec{w} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$. That is, if $\vec{w}$ is a linear combination of $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$.

     We can then proceed exactly as we did in problem 1. First, let A be the matrix whose columns are $\vec{v_1}, \vec{v_2}, \vec{v_3}$. Then we must solve the system $A\vec{x} = \vec{w}$. This can be done by row reduction, and one finds the reduced tow echelon form to be

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

     This is a consistent system, so there is a way to write $\vec{w}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$. Furthermore, this system has infinitely many solutions (consider the bottom row), so there are infinitely many ways to write $\vec{w}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

   - One quick way to reason that these three vectors are not linearly independent (that is, that they are linearly dependent) is to note that if they were linearly independent then there would be a unique way to write $\vec{w}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$. If we don't want to appeal to this general theorem, we can instead explicitly find a linear dependence – meaning an explicit way to write one as a linear combination of the others. To do this, we solve the system $x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} = \vec{0}$. This corresponds to solving the matrix equation $A\vec{x} = \vec{0}$. If we write out the augmented matrix and row reduce this, we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

     Notice that in comparison to the previous row reduction, only the final column is changed. When row reducing a matrix, the only numbers that can affect an entry are those in the same column as that entry. If this fact isn't clear, try row reducing both of these systems at the same time. You can do the same steps in both cases, and the left part of the augmented matrix will be row reduced in the same way in both cases.

     We now want to find a solution to this system. Well here's one: $x_1 = x_2 = x_3 = 0$. But this doesn't help us write one of the vectors in terms of the other two. Instead, we want to find a nonzero solution to this system. Indeed, let's translate this augmented matrix back into equations.

$$\begin{aligned} x_1 + 3x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

     Moving the x_3 to the right hand side yields

$$\begin{aligned} x_1 &= -3x_3 \\ x_2 &= -x_3 \end{aligned}$$

So that the solutions to this system are

$$\begin{pmatrix} -3x_3 \\ -x_3 \\ x_3 \end{pmatrix}$$

for any real number x_3. For example, take $x_3 = 1$. Then

$$\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

is a solution to this system, so $-3\vec{v_1} - \vec{v_2} + \vec{v_3} = \vec{0}$. One can check this equation by hand too (and checking work is always a good idea when doing computations!). This might not look like writing one equation in terms of the other two, but it gives us a specific relationship between all three. In particular, we rearrange this by moving $\vec{v_2}$ to the right hand side to get the equation $-3\vec{v_1} + \vec{v_3} = \vec{v_2}$. So we have written $\vec{v_2}$ in terms of $\vec{v_1}$ and $\vec{v_3}$.

- Succinctly, the system $A\vec{x} = \vec{u}$ has infinitely many solutions as the rank of A is 2, which is less than 3. More explicitly, let's start with the simplest possible example where we take $u = \vec{0}$. Is this in S? Well $\vec{0} = 0\vec{v_1} + 0\vec{v_2} + 0\vec{v_3}$ so yes, it is. What's another way to write this? We just found a way in the previous part where we showed that $-3\vec{v_1} - \vec{v_2} + \vec{v_3} = \vec{0}$. So we have two ways of writing $\vec{0}$ as a linear combination of the $\vec{v_i}$.

  How about for any $\vec{u}$ in S? Well any such vector is, by definition of S, a linear combination of the $\vec{v_i}$ so let's say $\vec{u} = a\vec{v_1} + b\vec{v_2} + c\vec{v_3}$ for some scalars a, b, c. To write this another way, let's use our new fancy way of writing the zero vector – $\vec{0} = -3\vec{v_1} - \vec{v_2} + \vec{v_3}$. If we add this to $\vec{u}$ nothing changes, since $\vec{u} + \vec{0} = \vec{u}$. But we can also compute this sum as

$$\begin{aligned} \vec{u} + \vec{0} &= (a\vec{v_1} + b\vec{v_2} + c\vec{v_3}) + (-3\vec{v_1} - \vec{v_2} + \vec{v_3}) \\ &= (a-3)\vec{v_1} + (b-1)\vec{v_2} + (c+1)\vec{v_3} \end{aligned}$$

  so we've written $\vec{u}$ as a linear combination of the $\vec{v_i}$ in a new way.

  A fascinating thing is happening here (but now I'm rambling beyond what is necessary for the problem, so what remains can be skipped). We could have done the same for any solution to $A\vec{x} = \vec{0}$ rather than the particular solution

$$\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

  we chose above. And in fact, this procedure describes all of the ways to write $\vec{u}$ as a linear combination of the $\vec{v_i}$. The solution set to $A\vec{x} = \vec{0}$ is a line in $\mathbb{R}^3$, so this line parameterizes all of the ways to write $\vec{u}$ as a linear combination of the $\vec{v_i}$. We can think of this line as all the ways to write $\vec{0}$ as a linear combination of the $\vec{v_i}$. So to describe all the ways to write $\vec{u}$ as a linear combination of the $\vec{v_i}$, we need to know *one* way to do so and we need to know all the ways to write $\vec{0}$ as a linear combination of the $\vec{v_i}$. This idea is a powerful one in linear algebra, where we can understand the solution set of a system of linear equations in terms of a single particular solution and the so–called "kernel" or "nullspace" of the matrix A. This is also helpful in solving differential equations, where one can vastly simplify the problem by finding a particular solution and then solving the "homogeneous" equation.

$\square$

3. *Proof.* One might be concerned about the variable h, but this is still the same method as in problem 1 and the first part of problem 2. We row reduce as usual, and just apply the same algebra we're doing to the numbers to the variable h. Let's do it explicitly.

   We want to solve the system $\vec{w} = x_1\vec{v_1} + x_2\vec{v_2}$. Writing this as an augmented matrix yields

$$\left(\begin{array}{cc|c} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & h \end{array}\right)$$

   Now let's row reduce this.

$$\left(\begin{array}{cc|c} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & h \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 7 & -14 \\ -1 & 2 & h \end{array}\right)$$
$$\xrightarrow{R_2/7} \left(\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 1 & -2 \\ -1 & 2 & h \end{array}\right)$$
$$\xrightarrow{R_3 + R_1} \left(\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 1 & -2 \\ 0 & -3 & h+3 \end{array}\right)$$
$$\xrightarrow{R_3 + 3R_2} \left(\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & h-3 \end{array}\right)$$
$$\xrightarrow{R_1 + 5R_2} \left(\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & -2 \\ 0 & 0 & h-3 \end{array}\right)$$

   This is in reduced row echelon form, and it corresponds to the system of equations

$$\begin{aligned} x_1 &= -7 \\ x_2 &= -2 \\ 0 &= h - 3 \end{aligned}$$

   So we've found x_1 and x_2 right? Well there's an issue – what is $h = 0$? Then that last equation is $0 = -3$, so the system is inconsistent which would tell us that there is no way to write $\vec{w}$ as a linear combination of $\vec{v_1}, \vec{v_2}$ in that case.

   So when do we have consistency? This will happen when the last equation is simply $0 = 0$, which happens precisely when $h - 3 = 0$. That is, when $h = 3$. So it is only when $h = 3$ that we can write $\vec{w}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$, and in that case we have $\vec{w} = -7\vec{v_1} - 2\vec{v_2}$. For all $h \neq 3$, this is not consistent.

   Geometrically, the span of $\vec{v_1}$ and $\vec{v_2}$ is a plane in $\mathbb{R}^3$ and $\vec{w}$ is some vector in $\mathbb{R}^3$. $\vec{w}$ depends on h but the plane does not. When $\vec{w}$ is in the span of $\vec{v_1}$ and $\vec{v_2}$ (that is, when $h = 3$) we have that $\vec{w}$ is on said plane. When $\vec{w}$ is not in the span of $\vec{v_1}$ and $\vec{v_2}$m the vector $\vec{w}$ is not on said plane. In that case, the three vectors are linearly independent. Intuitively, the fact that $\vec{w}$ is not on the plane spanned by $\vec{v_1}$ and $\vec{v_2}$ suggests that those three vectors yield three independent directions in three dimensional space.

By the way, in discussion I (attempted) to show a little graphic of this result. I don't yet know of a good way to share this nicely, but if you can use sagemath (say through cocalc as I did), here's the source code I used:

```
var('t, u')
scale_t = 3
scale_u = 1
@interact
def _(h=slider(-1,7, step_size=0.1)):
    plot = parametric_plot3d((t - 5 * u, 3*t - 8*u, -1 * t + 2 * u),
    (t, -1*scale_t, scale_t), (u, -1*scale_u, scale_u), aspect_ratio=[1,0.3,1])
    + parametric_plot((3*t, -5*t, h*t), (t, 0, 1), color='red', thickness = 5.0,
    aspect_ratio=[1,0.3,1])

    show(plot)
```

□

4. *Proof.* This is a key fact relating linear transformations and matrices. The proof is pure algebraic manipulation, resting on the following key identity.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1\vec{e_1} + \cdots + x_m\vec{e_m}$$

Indeed, let's see how this identity works in an example.

$$\begin{aligned} 1\vec{e_1} + 2\vec{e_2} + 0\vec{e_3} + 3\vec{e_4} &= 1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} \end{aligned}$$

We can think of this as saying that every vector is uniquely a linear combination of the standard basis vectors. To describe a position in m dimensional space, we must describe its components in the $x_1, x_2, \ldots, x_m$ directions. These directions are exactly represented by the $\vec{e_i}$.

Once we have this, we can do some matrix algebra to prove the result.

$$
\begin{aligned}
A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} &= A(x_1 \vec{e_1} + \cdots + x_m \vec{e_m}) \\
&= A(x_1 \vec{e_1}) + \cdots + A(x_m \vec{e_m}) \\
&= x_1 A(\vec{e_1}) + \cdots + x_m A(\vec{e_m}).
\end{aligned}
$$

□