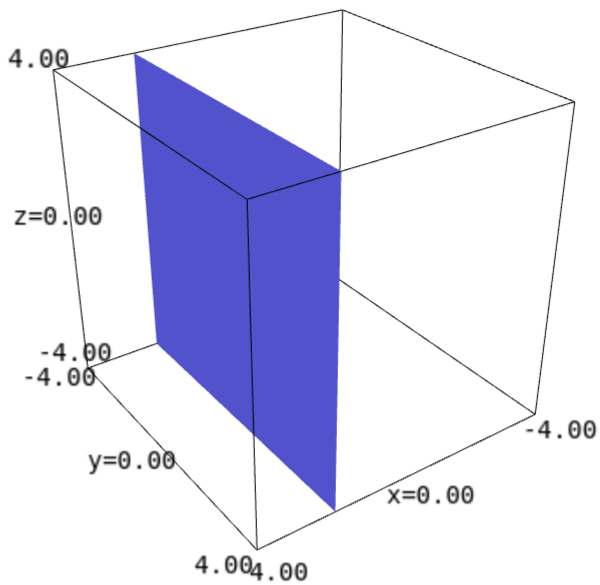
33A
Discussion Worksheet 1

1.

2. Before talking about the general case, let's consider the case $n = 3$ more closely. We are considering the solutions to the equation $ax + by + cz = d$ in $\mathbb{R}^3$. The change in notation from x_1, x_2, x_3 to x, y, z is just for the sake of aligning with my graphing software. Anyways, how can we describe the set of all such solutions geometrically?

   Let's first consider a special case like $a = 1$, $b = 0$, $c = 0$, $d = 2$. That is, we want the solutions to the equation $x = 2$. These are the vectors in $\mathbb{R}^3$ which have x coordinate equal to 2. Visually, the points in $\mathbb{R}^3$ satisfying this condition is as follows:

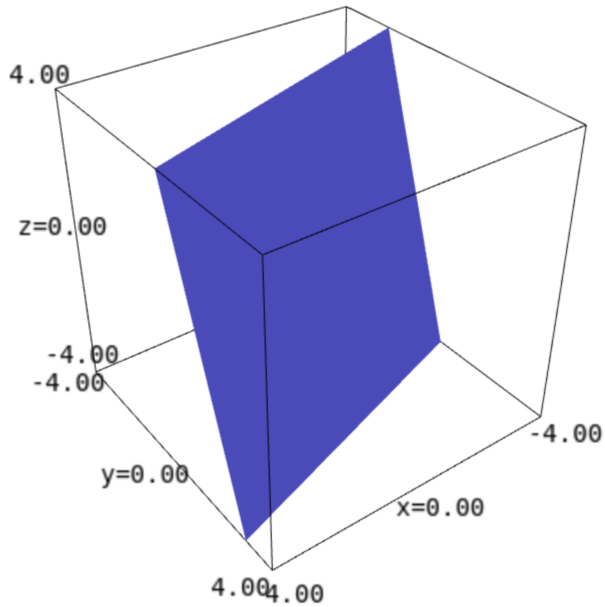


   Another example would be $x - 3y - z = -1$, which is plotted as

   If you were to try drawing these by hand, you might recall (say from a multivariable calculus class) that a plane given by the equation $ax + by + cz = d$ has normal vector parallel to $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

   So if we take the solutions to a single linear equation in $\mathbb{R}^2$ we get a line, which is one dimensional. If we take the solutions to a single linear equation in $\mathbb{R}^3$, we get a plane, which is two dimensional. That is, we ended up with something flat which is one dimension smaller than the number of variables. This is what happens in the general case as well. The set of solutions to a single linear equation in $\mathbb{R}^n$ is some generalization of lines and planes called a "hyperplane in $\mathbb{R}^n$". In $\mathbb{R}^2$ hyperplanes are lines and in $\mathbb{R}^3$ hyperplanes are planes. It will be flat and $n - 1$ dimensional. Furthermore, lines and planes both contain infinitely many points, and these hyperplanes will as well.

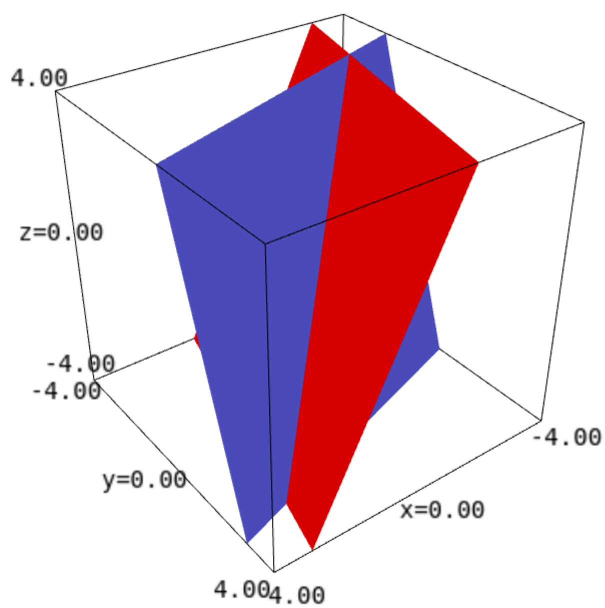
   It's impossible to graph these like with the planes, since we only have three dimensions at our disposal. We can occasionally use color or time or something like that to take the place of a fourth spatial dimension, but the result will never be as clean as the three dimensional drawings.



Now, what happens if we had multiple equations? This means that we are looking for those vectors in $\mathbb{R}^n$ which satisfy the first equation and the second equation and the third equation and . . . . Visually, this means that the set of solutions to a system of equations is the *intersection* of the solution sets of each individual equation. Therefore, it's the intersection of a bunch of hyperplanes. For example, consider the system

$$\begin{aligned} 4x - y + 2z &= 0 \\ x - 3y - z &= -1 \end{aligned}$$

The set of solutions to this system is given by the intersection of the solutions to the first equation with the solutions to the second equation. In the following plot, the first equation's solution set is in red and the second equation's solution set is in blue.

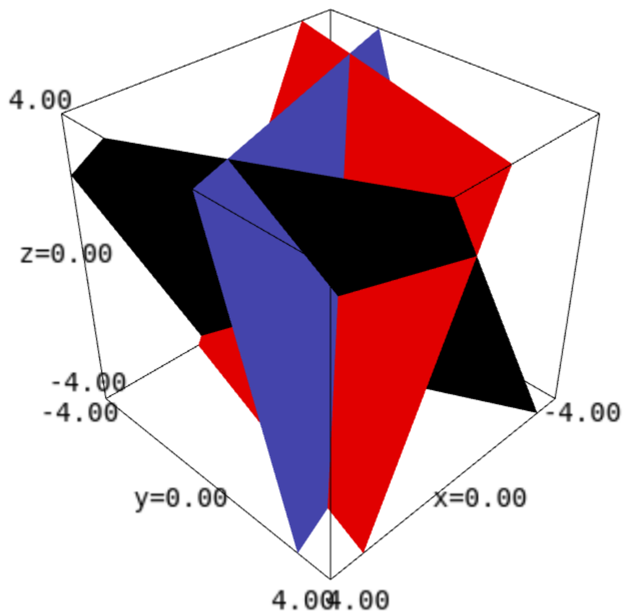


The intersection here is a line, which represents all of the solutions to the system.

Now say we added another equation to our system, like $4x+2y-2z=3$. So we are considering the system

$$\begin{aligned} 4x+2y-2x&=3\\ 4x-y+2z&=0\\ x-3y-z&=-1 \end{aligned}$$

Then we'd have to consider the intersection of three planes.



Here, the first equation's solution set is black, the second equation's solutions set is red, and the third equation's solution set is blue. It might be a bit messy, but there is a single point which lies in the intersection of all three planes. This is exactly the unique solution we found in problem 1!

With this in mind, let's think geometrically about why the set of solutions to a system has 0, 1, or infinitely many solutions. As above, the set of solutions to a system of linear equations is given by the intersection of hyperplanes. Based on the pictures above, if the intersection of hyperplanes is nonempty and not a point, it will be something like a line or bigger. But lines are infinite, so if we have more than 1 solution there are infinitely many. This is not a wholly rigorous argument, but it's a picture of why this fact holds.

As for how the cases may arise, I'll encourage you to try drawing some of your own examples too. For 0 solutions, we need an inconsistent system like

$$\begin{aligned} x&=0\\ y&=1\\ x+y&=0 \end{aligned}$$

This would correspond visually to hyperplanes that do not intersect at all. Can you draw 2 planes in $\mathbb{R}^3$ which don't intersect? 3 planes? 4 planes? Infinitely many planes?

For 1 solution, that's like the system in problem 1, which was plotted above. The intersection will be a single point.

For infinitely many solutions, we can consider the intersection of 2 planes in $\mathbb{R}^3$ like above. We can even consider the intersection of a plane with itself, which could correspond to a system like

$$\begin{aligned} 2x + 3y + z &= 0 \\ 4x + 6y + 2z &= 0 \end{aligned}$$

Here, the intersection is a line.

3. *Proof.* Indeed, the elementary row operations can all be undone. For example, let's say we swap rows 1 and 2 in our matrix A to get a matrix B. How can we reverse this? We just swap rows 1 and 2 again! Here's an example.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Swapping rows 1 and 2 yields

$$B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \\ 5 & 6 \end{pmatrix}$$

Swapping rows 1 and 2 again yields

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

In general, if we swap rows i and j to go from A to B, we can transform B back into A by swapping rows i and j again. So this row operation can be undone.

How about scaling a row? Suppose we transform our matrix A into a new matrix B by scalaing row i by a nonzero constant c. For example,

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

If we scale the first row by 2, we get the new matrix

$$B = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

How do we return to A? Well we divide the first row by 2! In other words, we multiply by $1/2$. Indeed, this gets right back to

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

So in general, when we scale row i by $c \neq 0$, we reverse this by scaling row i by $1/c$.

Finally, we get to adding a multiple of one row to another. Say we add cR_i to R_j. How can we undo this process? Let's go again with an example.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

If we add $2R_1$ to R_2, we get

$$B = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$$

So how do we get back to A? Let's subtract away what we added! We subtract $2R_1$ from R_2, that is, we add $-2R_1$ to R_2. This gets us back to

$$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

So in general, if we get B by adding cR_i to R_j (for $i \neq j$), we can get back to A by adding $-cR_i$ to R_j.

Therefore, all of the elementary row operations are reversible.

There's an interesting and important point to make here. This is closely related to the notion of matrix invertibility! Indeed, let's consider scaling one row by another. Consider

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Scaling row 1 by 2 yields

$$\begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}$$

Let's observe the following too:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}$$

This is the same as scaling the row! Furthermore,

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

And multiplying on the left by this new matrix corresponds to multiplying the first row by $\frac{1}{2}$. Hence, invertibility of this matrix corresponds precisely to the fact that we can reverse the row operation!

It turns out that all elementary row operations we can apply to A correspond to left multiplication by some matrix. And those matrices being invertible corresponds to reversing these row operations. I'd encourage trying to find what these matrices are yourself! As a hint, we get

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

by applying the row operation $R_1 \mapsto 2R_2$ to the identity matrix. This is precisely the row operation multiplying on the left by this matrix corresponds to. $\square$

4. *Proof.* The idea for this problem can be summed up quite quickly. If we apply a sequence of row operations, then as each row operation is reversible (via problem 3), we can reverse the sequence. The question then is, how do we write this down a bit more formally?

First, an example. Let's take

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$$

We'll apply some row operations. First, multiply row 2 by $\frac{1}{3}$. I notate this as $R_2 \mapsto \frac{1}{3}R_2$. This yields

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Next, we subtract row 2 from row 1, i.e. $R_1 \mapsto R_1 - R_2$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Let's do another operation, say $R_1 \leftrightarrow R_2$, by which I mean swap the rows.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Ok, now how do we reverse this? Our path to get here was $R_2 \mapsto \frac{1}{3}R_2$, $R_1 \mapsto R_1 - R_2$, and $R_2 \leftrightarrow R_1$. Let's do all this in reverse.

First, we reverse the swap by swapping again, $R_1 \leftrightarrow R_2$. This gets us from

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

to

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Next, we take $R_1 \mapsto R_1 + R_2$. This takes us to

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Finally, we scale row 2 via $R_2 \mapsto 3R_2$. This yields

$$\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$$

which is back to where we started.

So how do we do write this in general? We can try representing a sequence of row operations with a sequence of arrows

$$A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow A_n = B$$

with each arrow representing a single elementary row operation. For instance, this could be

$$A \xrightarrow{R_2 \mapsto \frac{1}{3}R_2} A_1 \xrightarrow{R_1 \mapsto R_1 - R_2} A_2 \xrightarrow{R_1 \leftrightarrow R_2} B$$

We represent the reversal as

$$A \xleftarrow{R_2 \mapsto 3R_2} A_1 \xleftarrow{R_1 \mapsto R_1 + R_2} A_2 \xleftarrow{R_1 \leftrightarrow R_2} B$$

This notation allows us to express the intuitive idea we had above, that we just reverse the operations step by step in the sequence.

We can also think of this in terms of the interpretation of row operations as matrix multiplication, which was discussed in problem 3. Indeed, a sequence of row operations applied to A results in the matrix

$$E_1 E_2 \dots E_n A$$

For whichever matrices E_i correspond to the elementary row operations, as discussed briefly in problem 3. To reverse these, we multiply by the inverse of these E_i, which is exactly the process of reversing their corresponding row operations.

$$E_n^{-1} \dots E_2^{-1} E_1^{-1} E_1 E_2 \dots E_n A = A$$

Note that in both interpretations, the order in which we apply the reversed operations must be reversed as well! □

5. *Proof.* We can apply a sequence of row operations to get from A to $\mathrm{rref}(A)$. This is just Gauss – Jordan elimination! So how do we get from $\mathrm{rref}(A)$ back to A? Well, this is exactly what we discussed in problem 4, taking $B = \mathrm{rref}(A)$. In short, we reverse the sequence row operations we did to get to $\mathrm{rref}(A)$ in turn. □

6. *Proof.* Here's the short answer: Two matrices of the same type are

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 1.483473933332498234 \\ 0 & 1 & \pi^{e+\sqrt{1.7779834793}} \end{pmatrix}$$

Two matrices of a different type are

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For this question, let's note that the condition of "being the same type" only concerns the leading 1s. Let's try building up some matrices from scratch. We'll start by placing some random choice of leading 1s down and leaving the rest of the matrix to be determined. For example,

$$\begin{pmatrix} 1 & * & * \\ * & 1 & * \end{pmatrix}$$

The askterisks $*$ are meant to be placeholders. We'll try to fill in these placeholders as we go along.

Remember that we are only concerned in this problem with matrices in reduced row echelon form. One condition of reduced row echelon form is that the first nonzero entry of each row has to be a 1. This automatically holds for the first row. For the second row, we haven't yet decided the entry to the left of our 1, so for this hypothetical matrix we're building to be in RREF, we'd need that entry to be a 0.

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \end{pmatrix}$$

Our hands were tied here. If we wrote anything but a 0 in the bottom left (aka the $(2,1)$ position), the matrix wouldn't have been in RREF.

Another condition for RREF is that in a column with a leading 1, every other entry must be 0. The first column has a leading 1, and the other entry in this column was forced to be 0 by what we wrote above too! The second column also has a leading 1, so we're forced to put a 0 everywhere else in that column. Namely, we must put a 0 in the $(1,2)$ position.

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \end{pmatrix}$$

How about the entries in this rightmost column? What should we put in those? Well in this context, it doesn't matter at all! Whatever numbers we choose to place in that rightmost column, we'll still get a matrix in RREF. Let's try some numbers as a sanity check.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

Yeah that's in RREF.

$$\begin{pmatrix} 1 & 0 & 1.483473933332498234 \\ 0 & 1 & \pi^{e+\sqrt{1.7779834793}} \end{pmatrix}$$

That's also in RREF.

These two matrices we just wrote down are an answer to this problem. They're both in RREF, they have leading 1s in exactly the same place, and they're distinct. In fact, we've done even more. We've figured out what *every* matrix of this "type" looks like. They're all of the generic form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \end{pmatrix}$$

with whatever choice of rightmost column we desire.

How about something of a different type? Let's start the same way we did for the first part of this problem, by just writing down some random leading 1s. Since we want to find something of a different type to the above matrices, let's choose these leadings 1s to be placed differently than those. For example,

$$\begin{pmatrix} 1 & * & * \\ * & * & 1 \end{pmatrix}$$

I'll be briefer, but if we do the same sort of analysis we'll see our hands are similarly tied. For instance, everything else in the first row has to be 0.

$$\begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \end{pmatrix}$$

And everything else in the last column has to be 0.

$$\begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This last entry can be chosen freely, so let's just pick anything to get something concrete.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a matrix in RREF that's of a different type to the ones we wrote above. $\square$

7. *Proof.* **The 2×2 matrices**

Here's the short answer: there are 4 types of 2×2 matrices in RREF, as follows.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where the asterisk $*$ represents a placeholder where we can put any number we want, as in problem 6.

But how did we get here? The key is that the only relevant data when determining if two matrices are of the same type is the position of their leading 1s. So in other words, we need to figure out in what ways we can put leadings 1s into a 2×2 RREF matrix.

So first off, there can be either 0, 1, or 2 leading 1s in a 2×2 matrix. Let's start with the case where there are 0 leading 1s. Well, a row without a leading 1 must contain only 0 if we want to be in RREF. So if there are no leading 1s, we can only have the 0 matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The next simplest case is when we have 2 leading 1s. That means that each row has to have a leading 1, as we only have 2 rows. Furthermore, the leading 1 in the second row has to be to the right of the leading 1 of the first row. But there are only 2 columns, so we're very constrained here. For instance, if we have

$$\begin{pmatrix} * & 1 \\ * & * \end{pmatrix}$$

Then there's no way to fit another leading 1 in the second row! Hence, the leading 1 in the first row *must* be in the top left corner.

$$\begin{pmatrix} 1 & * \\ * & * \end{pmatrix}$$

So where can the leading 1 in the second row be? It must be strictly to the right of the leading 1 in the first row, so our only choice is the bottom right corner.

$$\begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix}$$

When trying to write down a 2×2 matrix in RREF with 2 leading 1s, we had no choice but to put those leading 1s in the $(1, 1)$ and $(2, 2)$ position. Hence, there is only 1 type of 2×2 matrix in RREF with 2 leading 1s. In fact, we even know precisely what the placeholder values here can be. Everything else in a column with a leading 1 must be 0, so any RREF matrix of this type must be equal to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Alright, now we just have to figure out what types exist when we have only a single leading 1. If we only have one leading 1, then as there are two rows there must be some row without a leading 1. And in RREF, a row without a leading 1 must contain only zeroes. Furthermore, in RREF, the zero rows must be at the bottom. In this case, that means that the first row has a leading 1 somewhere, and the second row must be all zeroes.

$$\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$$

So where can we place leading 1s in the first row? Well we have two columns, and we can place one in either. If we place it in the first column, we get

$$\begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}$$

And this placeholder can be taken to be any number we want. On the other hand, if we put the leading 1 in the second column, we have

$$\begin{pmatrix} * & 1 \\ 0 & 0 \end{pmatrix}$$

But here, to be in RREF we need the leading 1 to be the first nonzero entry of the row. So this placeholder is forced to be 0. Then we are left with

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

To summarize, we split our analysis into three cases - when there are 0, 1, or 2 leading 1s. In each case, we followed the constraints imposed by RREF as far as we could. In the case of two leading 1s, we had to split into two further cases, one for each placement of the single leading 1 we had. With this, we have determined every type of 2×2 matrix in RREF.

The 2×3 matrices

Here's the short answer: there are four types of 3×2 matrices in RREF, as follows.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where the asterisk $*$ represents a placeholder where we can put any number we want, as in problem 6.

We could do the same sort of reasoning as in the 2×2 for this part, where we try writing down all possible placements of leading 1s and use the constraints imposed by RREF. If the work in wasn't wholly clear I encourage you to try doing the same sort of analysis here! But rather than repeat myself, I'll give a shortcut. We can reduce the 2×3 case to the 2×2 case!

You may indeed notice that the matrices I wrote down here are basically the same as those from the 2×2 case, just with an extra row of zeroes attached at the bottom. And indeed, there's a reason for this. We are considering here 3×2 matrices. How many leading 1s can there be? Certainly, no more than 3, as there can be at most one leading 1 per row. But there can also be at most one leading 1 per column, as everything else in a column which contains a leading 1 must be 0 in RREF. So there are in fact at most *two* leading 1s. This is a general idea, we ended up with at most the minimum of the number of rows and the number of columns.

Anyways, if we have at most two leading 1s and three rows, then there must be a row without a leading 1. A row without a leading 1 must contain all 0s to be in RREF. Also, any zero rows must be at the bottom of the matrix to be in RREF. So we see then that whatever type of matrix in RREF we write down, they must all look like

$$\begin{pmatrix} * & * \\ * & * \\ 0 & 0 \end{pmatrix}$$

The upper part here is just a 2×2 matrix. For this 3×2 matrix to be in RREF is the same as for that upper 2×2 matrix (the part with the asterisks) to be in RREF. This requires some thought, and I'd encourage you to think about why that's true! But given this fact, we have reduced our problem to finding all the types of RREF matrices that are 2×2, which is exactly what we did above. This saves us the work of having to redo this same analysis, and gives us the answer written at the start of this problem. $\square$